A FORBIDDEN SUBGRAPH CHARACTERIZATION PROBLEM AND A MINIMAL-ELEMENT SUBSET OF UNIVERSAL GRAPH CLASSES

by

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ABSTRACT

A FORBIDDEN SUBGRAPH CHARACTERIZATION PROBLEM AND A MINIMAL-ELEMENT SUBSET OF UNIVERSAL GRAPH CLASSES

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The *direct sum* of a finite number of graph classes $\mathcal{H}_1, ..., \mathcal{H}_k$ is defined as the set of all graphs formed by taking the union of graphs from each of the $\mathcal{H}_i$. The *join* of these graph classes is similarly defined as the set of all graphs formed by taking the join of graphs from each of the $\mathcal{H}_i$. In this paper we show that if each $\mathcal{H}_i$ has a forbidden subgraph characterization then the direct sum and join of these $\mathcal{H}_i$ also have forbidden subgraph characterizations. We provide various results which in many cases allow us to exactly determine the minimal forbidden subgraphs for such characterizations. As we develop these results we are led to study the minimal graphs which are *universal* over a given list of graphs, or those which contain each graph in the list as an induced subgraph. As a direct application of our results we give an alternate proof of a theorem of Barrett and Loewy concerning a forbidden subgraph characterization problem.

ACKNOWLEDGMENTS

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Contents

1 Introduction

Given a nonempty class $\mathcal{C}$ of graphs, a graph G is said to be $\mathcal{C}$-*free*, or to be an element of $\mathcal{G}(\mathcal{C})$, if none of its induced subgraphs is isomorphic to an element of $\mathcal{C}$. If a collection $\mathcal{H}$ of graphs is such that $\mathcal{H} = \mathcal{G}(\mathcal{C})$ for some collection $\mathcal{C}$ of graphs, we say that $\mathcal{H}$ has a *forbidden subgraph characterization.* Many important classes of graphs have forbidden subgraph characterizations. For example, if $\mathcal{C}$ is the collection of graphs homeomorphic to K_5 or $K_{3,3}$, then $\mathcal{G}(\mathcal{C})$ is the class of planar graphs [K]. If $\mathcal{C} = \{2K_2, C_4, C_5\}$, then $\mathcal{G}(\mathcal{C})$ is the class of split graphs, those whose vertex sets may be partitioned so as to form an independent set and a clique [FH]. Of recent note, Seymour et. al. have proved [CRST] that the perfect graphs, those for which the chromatic number equals the clique number in any given induced subgraph, are the $\mathcal{C}$-free graphs, where $\mathcal{C}$ denotes the class of cycles on $2n+1$ vertices ($n > 1$) and their complements. For a survey of results on forbidden-subgraph classes, see [BLS].

As we will note, the properties of many classes provide for the existence of forbidden subgraph characterizations, but such characterizations may be hard to find. In some cases, a given graph class may be described in terms of forbidden subgraphs in different ways. For example, in a paper recently submitted for publication [BL], Barrett and Loewy found one characterization for a class of graphs important in a matrix rank minimization problem, in terms of a list of forbidden subgraphs. The paper's referees found a different characterization, involving graphs formed by taking the union of elements from two different forbidden-subgraph classes. The question arises, then, of whether a forbidden subgraph characterization must exist for a class of graphs whose elements are unions of elements from forbidden-subgraph classes. Stated more precisely, if $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$ are arbitrary nonempty classes of nonempty graphs and we define their *direct sum*

$$\begin{aligned}&\mathcal{G}(\mathcal{H}_1) \oplus \mathcal{G}(\mathcal{H}_2) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)\\ &\quad = \{G \text{ a graph } \mid G = G_1 \cup G_2 \cup \cdots \cup G_k, \text{ where } G_i \in \mathcal{G}(\mathcal{H}_i) \text{ for } i = 1, 2, ..., k\},\end{aligned}$$

must $\mathcal{G}(\mathcal{H}_1) \oplus \mathcal{G}(\mathcal{H}_2) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$ have a forbidden subgraph characterization, and if so, what is it?

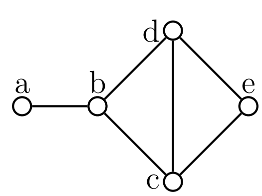
In this thesis we will show that $\mathcal{G}(\mathcal{H}_1) \oplus \mathcal{G}(\mathcal{H}_2) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$ does indeed have a forbidden subgraph characterization, and we will consider in depth the special case of this problem in which each $\mathcal{H}_i$ consists of a single nonempty graph H_i. We will give a "best possible" forbidden subgraph characterization of this class when each H_i is connected, and will discuss various simplifications of a forbidden subgraph characterization in the case when one or more H_i are disconnected. We finish by applying our results to various examples, including the characterizations of Barrett/Loewy and the referees, showing that the two are equivalent.

2 Definitions and Elementary Results from Graph Theory

This section contains many definitions and basic results from graph theory that will be important in the sections ahead. For further definitions and results, see [M] or [W].

A *graph* $G = (V, E)$ is a pair consisting of a *vertex set* V, also denoted $V(G)$, and an *edge set* E, also denoted $E(G)$. The vertex set, whose elements are called *vertices*, is usually defined to be a nonempty finite set; though our vertex sets will always be finite, for our purposes it will be useful at times to allow the vertex set to be empty. Such a graph $G = (\emptyset, \emptyset)$ is called an *empty*, or *null* graph. Adopting a convention followed in [M], we denote by $A^{(2)}$ the collection of all two-element subsets of a set A. Then the edge set $E(G)$ can be any subset of $V(G)^{(2)}$. As a notational convenience, we may denote an edge of G by uv or vu instead of $\{u, v\}$. Graphs thus defined are called *simple* graphs; graphs that are not simple (i.e., that allow loops and multiple edges) are treated in [B].

A graph may be represented pictorially by drawing a circle or point corresponding to each vertex of G and drawing an arc joining the two points corresponding to vertices u and v if uv is an edge of G. For example, let $V = \{a, b, c, d, e\}$ and $E = \{ab, bc, bd, cd, ce, de\}$. Then we can draw $G = (V, E)$ as the following:



Of course, by placing vertices in different relative positions we may draw infinitely many different representations of any given graph.

Given two graphs G and H, an *isomorphism* from G to H is a bijective map $\varphi : V(G) \to V(H)$ such that $\varphi(u)\varphi(v)$ is an edge of H if and only if uv is an edge of G. If an isomorphism exists from $V(G)$ to $V(H)$, we write $G \cong H$. The binary relation $\cong$ is an equivalence relation, and we call the associated equivalence classes *isomorphism classes.* An isomorphism $\varphi : V(G) \to V(G)$ is called an *automorphism.* We say that G is *vertex-transitive* if for any vertices $u, v \in V(G)$ there exists an automorphism $\phi : V(G) \to V(G)$ such that $\phi(u) = v$.

Given two vertices $u, v \in V(G)$, we say that u and v are *adjacent*, or that they are *neighbors*, if $uv \in E(G)$. For any vertex $v \in V(G)$, define the *neighborhood of v in G*, denoted $N_G(v)$, or $N(v)$, to be the set of all vertices of G that are adjacent to v.

A *subgraph* of G is a graph $H = (W, F)$, where $W \subseteq V(G)$ and $F \subseteq E(G) \cap W^{(2)}$. We say that H is a *proper* subgraph of G if W is properly contained in $V(G)$. If $F = E(G) \cap W^{(2)}$ we say that H is an *induced* subgraph of G, and we write $H = G[W]$. If G is a graph and $v \in V(G)$, we denote by $G - v$ the graph $G[V(G)\backslash\{v\}]$, and we say that $G - v$ is the graph obtained from G by *deleting* v. If J is isomorphic to an induced subgraph of G, we often say that G *induces* J, or that J *is induced in* G. Throughout the text we will use the words "contain" and "induce" interchangeably.

The *complement* of $G = (V, E)$, denoted G^c, is the graph $(V, V^{(2)}\backslash E)$.

The *union* of a finite number of graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots,$

$G_k = (V_k, E_k)$, where the V_i are all disjoint, is denoted by $G_1 \cup \cdots \cup G_k$, or by $\bigcup_{i=1}^{k} G_i$, and is the graph $(V_1 \cup \cdots \cup V_k, E_1 \cup \cdots \cup E_k)$. We often write nG to denote the graph $G \cup G \cup \cdots \cup G$ (n times).

In a graph $G = (V, E)$, a set $W \subseteq V$ is said to be a *clique* if the vertices of W are all pairwise adjacent, and an *independent set* if the vertices of W are all pairwise nonadjacent. A subgraph of G is said to be *complete* if the vertex set of the subgraph forms a clique.

A *path* in a graph is an alternating sequence of vertices and edges $(v_0, e_1, v_1, e_2, ..., e_k, v_k)$ such that v_{i-1} and v_i are adjacent for all $i = 1, 2, ..., k$, no vertex appears more than once in the sequence, and $e_i = v_{i-1}v_i$ for $i = 1, 2, ..., k$. We often simplify notation by writing the path as $v_0 - v_1 - \cdots - v_k$. We call v_0 and v_k the *endpoints* of the path, and $v_1, v_2, ..., v_{k-1}$ the *intermediate* vertices of the path. We define the *length* of the path to be k, the number of edges in the sequence. We say that a graph G is a path if there exists a path in G containing each vertex and each edge of the graph. The *distance between u and v in G*, denoted $d(u, v)$, or $d_G(u, v)$, is the length of the shortest path in G having u and v as its endpoints, if one exists, and is defined to be infinite otherwise. We consider the sequence (v) as a path of length zero, and hence $d(v, v) = 0$ for any $v \in V(G)$.

Proposition 2.1. *If $u = p_0 - p_1 - \cdots - p_{k-1} - p_k = v$ is a shortest path from u to v in G, then $G[\{p_0, ..., p_k\}]$ is a path.*

Proof: If $G[\{p_0, ..., p_k\}]$ is not a path, then there exists some edge $p_ip_j \in E(G)$ with $i < j$ and $j - i > 1$. Then

$$p_0 - p_1 - \cdots - p_{i-1} - p_i - p_j - p_{j+1} - \cdots - p_k$$

is a path of length $i + (k - j) + 1 = k - (j - i) + 1 < k - 1 + 1 = k$ and is therefore a shorter path from u to v, a contradiction. Hence $G[\{p_0, ..., p_k\}]$ is a path. $\square$

A *walk* is an alternating sequence of vertices and edges $(v_0, e_1, v_1, e_2, ..., e_k, v_k)$ such that v_{i-1} and v_i are adjacent for all $i = 1, 2, ..., k$, and $e_i = v_{i-1}v_i$ for $i = 1, 2, ..., k$. Notation similar to that used for paths is also used for walks.

Proposition 2.2. *[W, page 21] Every walk contains a path as a subsequence.*

A *component* C of G is a maximal nonempty induced subgraph of G such that for any two vertices $u, v \in V(C)$, there exists a path in G whose vertices all belong to $V(C)$ and whose endpoints are u and v. Alternatively, a component is a maximal nonempty induced subgraph C such that the distance between any two vertices of C is finite. A graph is said to be *connected* if it consists of at most one component. A *cutvertex* of G is a vertex $v \in V(G)$ such that $G - v$ has more components than G does.

Proposition 2.3. *[W, page 29] Every graph on $n \geq 2$ vertices has at least two vertices that are not cutvertices.*

Proposition 2.4. *[H, pages 27, 30] The following statements are equivalent:*

(1) v is a cutvertex of G;

(2) There exist vertices u, w of G distinct from v such that v lies on every path from u to w;

(3) There exist neighbors u, w of v in G such that v lies on every path from u to w.

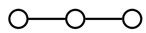
Proposition 2.5. *[CEJZ] For any graph $G = (V, E)$, given a vertex $u \in V$, let v be a vertex of G such that $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. Then v is not a cutvertex of G.*

A *dominating* vertex of G is a vertex that is adjacent to every other vertex in G. A *pendant* vertex is a vertex having exactly one neighbor. It is immediate from Proposition 2.4 that a pendant vertex is not a cutvertex.

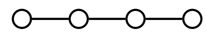
Given an arbitrary nonempty class $\mathcal{C}$ of nonempty graphs, a graph G is said to be *$\mathcal{C}$-free* if G induces none of the elements of $\mathcal{C}$. If $F \in \mathcal{C}$ we may write that G is *F-free*, or if $\mathcal{C} = \{F_1, ..., F_k\}$, we often write that G is *$(F_1, ..., F_k)$-free*. We let $\mathcal{G}(\mathcal{C})$ denote the class of graphs that induce no member of $\mathcal{C}$, and we write $\mathcal{G}(A_1, ..., A_j)$ in place of $\mathcal{G}(\{A_1, ..., A_j\})$. We note now that the empty graph is an element of $\mathcal{G}(\mathcal{C})$ for every class $\mathcal{C}$ of nonempty graphs. If $\mathcal{H}$ is a class of graphs, we say that F is a *forbidden subgraph* for $\mathcal{H}$ if no element of $\mathcal{H}$ induces F. If $\mathcal{H} = \mathcal{G}(\mathcal{C})$ for some class $\mathcal{C}$ of graphs, we say that $\mathcal{H}$ has a *forbidden subgraph characterization*.

We end this section by cataloguing a few graphs by name that we will use throughout the thesis. They are as follows:

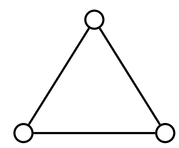
P_n - the path on n vertices.
K_n - the complete graph on n vertices.
$K_{n_1,...,n_k}$ - the complete k-partite graph with partite sets of orders $n_1, ..., n_k$ (alternatively, the complement of $K_{n_1} \cup \cdots \cup K_{n_k}$).
S_n - the star on n vertices (alternatively, $K_{1,n-1}$).
C_n - the cycle on n vertices.



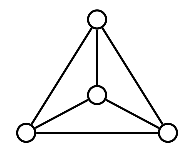
P_3



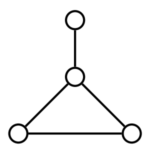
P_4



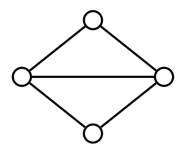
K_3 (C_3, triangle)



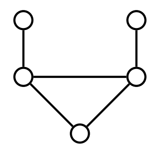
K_4



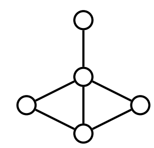
paw



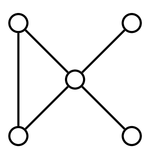
diamond



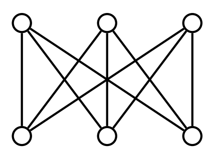
bull



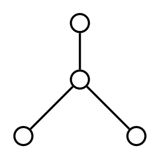
dart



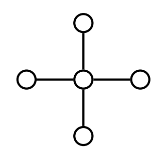
$\ltimes$



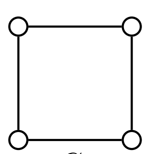
$K_{3,3}$



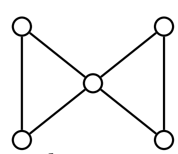
S_4 ($K_{1,3}$)



S_5 ($K_{1,4}$)



C_4



bowtie

3 The Existence of a Forbidden Subgraph Characterization

To begin this section, we first define an important property of certain graph classes and then prove a result that allows us to resolve our question about the existence of a forbidden subgraph characterization for $\mathcal{G}(\mathcal{H}_1) \oplus \mathcal{G}(\mathcal{H}_2) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$.

Definition 1. A class $\mathcal{C}$ of graphs is said to be *hereditary* if $G \in \mathcal{C}$ implies that every induced subgraph of G also belongs to $\mathcal{C}$.

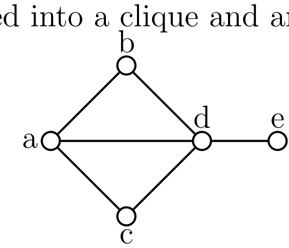
The following lemma was proved in Greenwell et. al. in [GHK]. We give a proof of it here, as the proof will be instructive in what is to follow.

Lemma 3.1. *A graph class $\mathcal{C}$ has a forbidden subgraph characterization if and only if it is hereditary.*

Proof: Suppose $\mathcal{C} = \mathcal{G}(\mathcal{D})$ for some class $\mathcal{D}$ of graphs. Let G be an arbitrary graph in $\mathcal{C}$. Then for any induced subgraph H of G and any induced subgraph I of H, I is also an induced subgraph of G and is therefore not isomorphic to any element of $\mathcal{D}$. Then by definition we have $H \in \mathcal{G}(\mathcal{D})$, so $\mathcal{C}$ is a hereditary class.

Suppose conversely that $\mathcal{C}$ is an arbitrary hereditary class of graphs, and define $\mathcal{D}$ to be the set of all graphs not in $\mathcal{C}$. Then $\mathcal{C}$ and $\mathcal{D}$ partition the class of graphs. Let G be an arbitrary element of $\mathcal{C}$. Then every induced subgraph of G belongs to $\mathcal{C}$, so no induced subgraph of G is isomorphic to an element of $\mathcal{D}$; thus $\mathcal{C} \subseteq \mathcal{G}(\mathcal{D})$. Now let H be an arbitrary element of $\mathcal{G}(\mathcal{D})$. Then no induced subgraph of H is isomorphic to an element of $\mathcal{D}$; in particular H itself is not an element of $\mathcal{D}$. Then by definition we have that $H \in \mathcal{C}$, so $\mathcal{G}(\mathcal{D}) \subseteq \mathcal{C}$, and we conclude that $\mathcal{C} = \mathcal{G}(\mathcal{D})$. Thus $\mathcal{C}$ has a forbidden subgraph characterization. $\square$

Example 3.1. As mentioned in the introduction, a *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set. For example,



is a split graph, since $\{a, d\}$ is a clique and $\{b, c, e\}$ is an independent set. We claim that the class of split graphs is a hereditary class. Suppose G is any split graph, and let V_1, V_2 be a partition of $V(G)$ such that V_1 is a clique and V_2 is an independent set. Then for any induced subgraph H of G, we have that $V_1 \cap V(H), V_2 \cap V(H)$ is a partition of $V(H)$ into a clique and an independent set. Hence H is split, and the class of split graphs is a hereditary class. By Lemma 3.1, the class of split graphs thus has a forbidden subgraph characterization. $\square$

Example 3.2. A graph G is said to be *eulerian* if there exists some sequence $v_0, v_1, ..., v_n$ of vertices of G such that $v_n = v_0$ and $v_1v_2, v_2v_3, ..., v_{n-1}v_n$ is a sequence of edges of G such that each edge of G appears exactly once in the sequence. Let $\mathcal{C}$ denote the class of eulerian graphs. Then $\mathcal{C}$ is not a hereditary class. For example, in Figure 1, graph (i) is eulerian, while graph (ii) is an induced subgraph of (i) and is not eulerian. Hence, by Lemma 3.1, the class of eulerian graphs does not have a forbidden subgraph characterization. $\square$

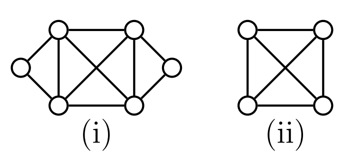


Figure 1: (i) eulerian; (ii) non-eulerian

Applying Lemma 3.1 to the problem at hand, we obtain the following:

Theorem 3.2. *For any nonempty classes $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$ of nonempty graphs, the class $\mathcal{G} = \mathcal{G}(\mathcal{H}_1) \oplus \mathcal{G}(\mathcal{H}_2) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$ has a forbidden subgraph characterization.*

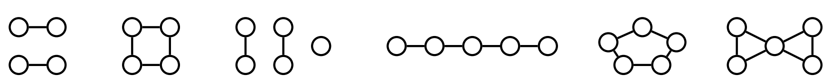
Proof: We will show that $\mathcal{G}$ is a hereditary class and then apply Lemma 3.1.

Suppose $G \in \mathcal{G}$ is arbitrary, and let $G_1, G_2, ..., G_k$ denote vertex-disjoint subgraphs of G such that $G_i \in \mathcal{G}(\mathcal{H}_i)$ for $i = 1, ..., k$, and $G = G_1 \cup \cdots \cup G_k$. Also let $V_i = V(G_i)$ for $i = 1, ..., k$. If G' is any induced subgraph of G, then there exists $V' \subseteq V(G)$ such that $G' = G[V']$. Define $G'_i = G_i[V(G_i) \cap V']$ for $i = 1, ..., k$. Then $G'_i \in \mathcal{G}(\mathcal{H}_i)$ for all i, since G'_i is an induced subgraph of G_i, which is an element of the hereditary class $\mathcal{G}(\mathcal{H}_i)$. Now

$$G' = G'_1 \cup \cdots \cup G'_k \in \mathcal{G}(\mathcal{H}_1) \oplus \mathcal{G}(\mathcal{H}_2) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k) = \mathcal{G},$$

and we see that $\mathcal{G}$ is a hereditary class. Then by Lemma 3.1, $\mathcal{G}$ has a forbidden subgraph characterization. $\square$

Note that while Lemma 3.1 guarantees the existence of a forbidden subgraph characterization for a class such as $\mathcal{G}$, its proof suggests only a very crude way of obtaining one—namely, taking as the set of forbidden subgraphs the set of all graphs not in the class. This forbidden set is necessarily infinite, and thus in most cases unwieldly to work with; and, in fact, much smaller lists of forbidden subgraphs may serve to characterize the graph class. To return to Example 3.1 again, we note that each of the following six graphs is not contained in, and hence is a forbidden subgraph for, the class of split graphs.



Thus, the split graphs form a subset of $\mathcal{G}(2K_2, C_4, 2K_2 \cup K_1, P_5, C_5, \text{bowtie})$. Can we conclude, though, that the class of split graphs *equals* the set $\mathcal{G}(2K_2,\, C_4,\, 2K_2 \cup K_1,\, P_5,$

C_5, bowtie)? And if so, is each graph of these six necessary as a forbidden subgraph to characterize the split graphs, or may we omit any?

We note that each of $2K_2 \cup K_1, P_5$, and the bowtie graph induce $2K_2$, so if a graph is $2K_2$-free, it must be free of these graphs as well. In general, we have the following:

Observation 3.3. If $\mathcal{C}$ is an arbitrary nonempty collection of graphs and $\mathcal{D} \subseteq \mathcal{C}$ is such that every graph in $\mathcal{C}$ induces an element in $\mathcal{D}$, then $\mathcal{G}(\mathcal{C}) = \mathcal{G}(\mathcal{D})$.

This reduces our list above to the graphs $2K_2, C_4$, and C_5. Do these graphs form a forbidden-subgraph collection large enough to characterize the split graphs, or must we include more forbidden subgraphs? In answer to this question, Foldes and Hammer showed [FH] that the split graphs are exactly the class of $(2K_2, C_4, C_5)$-free graphs, so our original infinite list of forbidden subgraphs for the split graphs (every graph that is not a split graph) can be reduced to this list of three graphs.

This will be the focus of our study for the remainder of the thesis: given arbitrary nonempty graphs $H_1, H_2, ..., H_k$, in what ways can we reduce our list of forbidden subgraphs of $\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k)$ to something more convenient to work with, and what properties will the graphs in our reduced list possess?

4 The Classes $\mathcal{CU}$ and $\mathcal{CU}^*$

4.1 Definitions and Elementary Results

Before returning to the problem at hand, we establish some results concerning induced subgraphs. In particular, we introduce a class of graphs that will be important to our solution.

Definition 2. Given a nonempty collection $\mathcal{H}$ of graphs, a graph G is said to be *universal* over $\mathcal{H}$ if every graph in $\mathcal{H}$ is isomorphic to an induced subgraph of G.

The concept of a universal graph was first introduced by Rado in [R]. In his and others' papers, a universal element of a graph class is defined to be an element of the class that contains all other elements of the class as subgraphs. With such a definition, the graphs considered usually have infinite vertex sets. Note that here we do not talk about a universal element of a class; by our definition a graph G need not be an element of $\mathcal{H}$ to be universal over it.

Definition 3. Given a collection $\mathcal{C} = \{A_1, A_2, ..., A_j\}$ of graphs, define $\mathcal{CU}(\mathcal{C}) = \mathcal{CU}(A_1, A_2, ..., A_j)$ to be the set of all connected graphs containing $A_1, A_2, ..., A_j$ as induced subgraphs, i.e., the set of all connected graphs that are universal over $\mathcal{C}$. When the context makes clear what is meant, we will simply call this class $\mathcal{CU}$.

Note that this class is necessarily infinite. A subset of this class that will be important to us later is the following:

Definition 4. With $\mathcal{C}$, $A_1, A_2, ..., A_j$, and $\mathcal{CU}$ as defined above, define $\mathcal{CU}^*(\mathcal{C}) = \mathcal{CU}^*(A_1, A_2, ..., A_j)$ to be a subset of $\mathcal{CU}$ with the following properties:

(1) Every graph in $\mathcal{CU}$ contains at least one graph of $\mathcal{CU}^*(\mathcal{C})$ as an induced subgraph.

(2) No graph in $\mathcal{CU}^*(\mathcal{C})$ is induced by any other graph in $\mathcal{CU}^*(\mathcal{C})$.

We call $\mathcal{C}$ a *generating set* and its elements *generators* for the class $\mathcal{CU}^*(\mathcal{C})$. When the context makes clear what is meant, we omit mention of the generating set and denote the class $\mathcal{CU}^*(\mathcal{C})$ simply by $\mathcal{CU}^*$.

To see that such a nonempty subset exists, consider the following: Since $\mathcal{CU}(A_1, A_2, ..., A_j)$ is nonempty, there are graphs in it of smallest order n. Since no proper induced subgraph of any of these graphs belongs to $\mathcal{CU}$, in order to satisfy property (1) of $\mathcal{CU}^*$, we must have that every one of these graphs belongs to $\mathcal{CU}^*$. We proceed inductively on the number of vertices in the graphs in $\mathcal{CU}$. For each $B \in \mathcal{CU}$ on s vertices, we check to see that no proper induced connected subgraph of B is isomorphic to any graph already included in $\mathcal{CU}^*$. If B does not induce any element of $\mathcal{CU}^*$, we append B to $\mathcal{CU}^*$; if it does, we discard B.

Proposition 4.1. *A graph $H \in \mathcal{CU}(\mathcal{C})$ belongs to $\mathcal{CU}^*(\mathcal{C})$ if and only if no proper induced subgraph of it belongs to $\mathcal{CU}(\mathcal{C})$.*

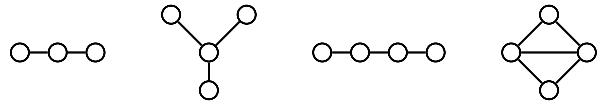
Proof: Suppose $H \in \mathcal{CU}^*$. Then if a proper induced subgraph J of H belonged to $\mathcal{CU}$, by the definition of $\mathcal{CU}^*$ there would be some induced subgraph K of J lying in $\mathcal{CU}^*$. But certainly K is also a proper induced subgraph of H, contradicting the definition of $\mathcal{CU}^*$. Hence, no proper induced subgraph of H belongs to $\mathcal{CU}$.

Suppose conversely that no proper induced subgraph of $H \in \mathcal{CU}$ belongs to $\mathcal{CU}$. By the definition of $\mathcal{CU}^*$, H must have an induced subgraph belonging to $\mathcal{CU}^*$, a subset of $\mathcal{CU}$. Since no proper induced subgraph will do, H itself must be an element of $\mathcal{CU}^*$. $\square$

Corollary 4.2. $\mathcal{CU}^*(\mathcal{C})$ *is unique.*

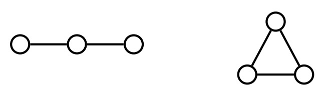
Proof: By Proposition 4.1, and noting that $\mathcal{CU}^*$ is, by definition, a subset of $\mathcal{CU}$, the elements of $\mathcal{CU}^*$ are uniquely determined. $\square$

Example 4.1. Consider the following four graphs, which all belong to $\mathcal{CU}(K_2^c)$:

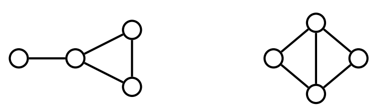


Note that each of these graphs is connected and induces K_2^c. However, applying Proposition 4.1, we see that none lies in $\mathcal{CU}^*$ except the first graph, P_3. In fact, we show that $\mathcal{CU}^* = \{P_3\}$: Let G be an arbitrary graph in $\mathcal{CU}^*$. Since G induces K_2^c, there exist two nonadjacent vertices u and v in G. Since G is connected, there exists a shortest path joining u and v. Since u and v are not adjacent, this path must contain at least 3 vertices and, by Proposition 2.1, thus induce P_3. Since $P_3 \in \mathcal{CU}$ and $G \in \mathcal{CU}^*$, by Proposition 4.1 we see that P_3 is not a proper induced subgraph of G; hence, $G = P_3$, and our proof is complete. $\square$

Example 4.2. Consider the graphs P_3 and K_3:



We show that $\mathcal{CU}^*(P_3, K_3)$ consists of the graphs

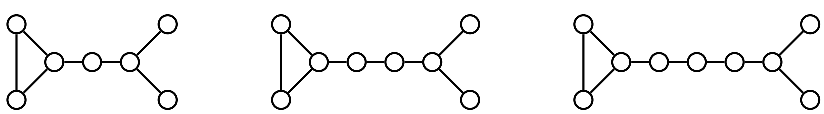


i.e., $\mathcal{CU}^*(P_3, K_3) = \{\text{paw}, \text{ diamond}\}$. Certainly the paw and the diamond belong to $\mathcal{CU}^*$ by Proposition 4.1, since no proper induced subgraph of them is a connected graph inducing both P_3 and K_3. Next, suppose that H is a graph in $\mathcal{CU}^*$ that is isomorphic to neither the paw nor the diamond. Then it contains neither of these as an induced subgraph, by the definition of $\mathcal{CU}^*$. Let u, v, w be the vertices of an induced subgraph of H isomorphic to K_3. Let

$$N_i = \{t \in V(H) \mid \min\{d(t,u), d(t,v), d(t,w)\} = i\}$$

for each $i \in \mathbb{N}$. Since $H \ncong K_3$, certainly N_1 must be nonempty. Let a be a vertex in N_1. Then a is adjacent to at least one vertex in $\{u, v, w\}$, and in order to not induce a paw or a diamond on these four vertices, a must be adjacent to each of u, v, and w. If $|N_1| \geq 2$, let b, c be any two vertices of N_1. Then b and c are each adjacent to each of u, v, w, and if b and c are not mutually adjacent, then $\{b, c, u, v\}$ induces a diamond, a contradiction. Hence, $N_1 \cup \{u, v, w\}$ induces a complete subgraph. Now if N_i is nonempty for any $i \geq 2$, then certainly there exists a vertex $d \in N_2$. Then d is adjacent to some vertex $f \in N_1$ and is not adjacent to any vertex in $\{u, v, w\}$. Then $\{d, f, u, v\}$ induces a paw, a contradiction. Then for all $i \geq 2$, we have that N_i is empty. This implies that the vertices of the induced P_3 all lie within $N_1 \cup \{u, v, w\}$, a contradiction, since the latter is a clique. Hence, $\mathcal{CU}^*$ contains no graphs distinct from the paw or the diamond, and our proof is complete. $\square$

Example 4.3. The class $\mathcal{CU}^*(A_1, A_2, ..., A_j)$ need not be finite; consider, for example, $\mathcal{CU}^*(S_4, K_3)$. By Proposition 4.1, each of the following graphs lies in $\mathcal{CU}^*$:



Note that if we continue to lengthen the graph by adding vertices to the path joining the copies of K_3 and S_4, we continue to create graphs belonging to $\mathcal{CU}^*$. Since we can continue this indefinitely, we conclude that $\mathcal{CU}^*(K_3, S_4)$ is infinite. $\square$

4.2 Special Cases

We see from the previous examples that $\mathcal{CU}^*$-sets vary much in the structure of their graphs, depending on the graphs $A_1, A_2, ..., A_j$. We can obtain much information about the structure of graphs in certain $\mathcal{CU}^*$-sets if we put further conditions on the A_i. This section treats a few special cases.

4.2.1 Connected graphs

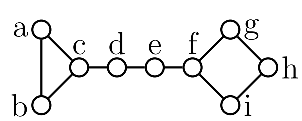
Definition 5. In this and the following subsection, let $H = (V, E)$ be an arbitrary element of $\mathcal{CU}^*(A_1, ..., A_j)$, where each A_i is a connected graph, and let $V_1, ..., V_j \subseteq V(H)$ be such that $H[V_i] \cong A_i$ for $i = 1, 2, ..., j$. In addition, define $X = V \backslash \left(\bigcup_{i=1}^{j} V_i \right)$.

Observation 4.3. If $|V| > \sum_{i=1}^{j} |V_i|$, then X is nonempty.

Proposition 4.4. *If $x \in X$ then x is a cutvertex in H, and for each V_i there exists some $k \in \{1, ..., j\}$ such that the component of $H - x$ in which V_i lies does not induce A_k; hence V_i and V_k lie in distinct components in $H - x$ and $V_i \cap V_k = \emptyset$.*

Proof: The union $\bigcup_{i=1}^{j} V_i \subseteq V \backslash \{x\}$, so $H - x$ contains induced $A_1, A_2, ..., A_j$. Since $H \in \mathcal{CU}^*(A_1, A_2, ..., A_j)$, by Proposition 4.1 we must have that $H - x$ is disconnected, so x is a cutvertex of H. Furthermore, suppose that for some V_i the component of $H - x$ containing V_i induces A_k for all $k \in \{1, 2, ..., j\}$. Then that component of $H - x$ is a connected proper induced subgraph of H that induces $A_1, A_2, ..., A_j$, a contradiction to Proposition 4.1. Hence for every $x \in X$ and every V_i there is a V_k such that the removal of x separates V_i and V_k. $\square$

Example 4.4. By Proposition 4.1 it is easy to verify that



is an element of $\mathcal{CU}^*(K_3, P_4, C_4)$. We are required to let $V_1 = \{a, b, c\}$ and $V_3 = \{f, g, h, i\}$, and there are multiple possibilities for V_2. Suppose $V_2 = \{e, f, g, h\}$. Then $X = \{d\}$, and as proved above, d is a cutvertex of G. Furthermore, for each V_i, $i = 1, 2, 3$, there is a V_k such that V_i, V_k are disjoint, since V_1 is disjoint from both V_2, V_3. $\square$

4.2.2 Two connected graphs

When we let $j = 2$ in the previous subsection, we can obtain even stronger results about the graphs in $\mathcal{CU}^*$. For example, the following results are natural corollaries of Proposition 4.4.

Corollary 4.5. *If $H \in \mathcal{CU}^*(A_1, A_2)$ and $|V(H)| \geq |V(A_1)| + |V(A_2)|$, then V_1 and V_2 are disjoint.*

Proof: Suppose that $V_1 \cap V_2 \neq \emptyset$. Then

$$|V_1 \cup V_2| = |V_1| + |V_2| - |V_1 \cap V_2| < |V(A_1)| + |V(A_2)| \leq |V(H)|,$$

so X is nonempty. Let x be a vertex of X. By Proposition 4.4, $H[V_1]$ and $H[V_2]$ lie in separate components in $H - x$, a contradiction to our supposition that $V_1 \cap V_2 \neq \emptyset$. Hence we conclude that $V_1 \cap V_2 = \emptyset$. $\square$

Corollary 4.6. *If $H \in \mathcal{CU}^*(A_1, A_2)$ and $|V(H)| > |V(A_1)| + |V(A_2)|$, then there exists no edge uv in H with $u \in V_1$, $v \in V_2$.*

Proof: Since $|V(H)| > |V(A_1)| + |V(A_2)|$, by Observation 4.3 we see that X is nonempty. Then by Proposition 4.4, $H[V_1]$ and $H[V_2]$ lie in different components in $H - x$ for any $x \in X$; hence there can be no edge joining a vertex in V_1 with one in V_2. $\square$.

Proposition 4.7. *If $H \in \mathcal{CU}^*(A_1, A_2)$ and X is nonempty, then $H[X]$ is a path with one endpoint having at least one neighbor in V_1, and the other endpoint having at least one neighbor in V_2.*

Proof: Given any vertex $a \in V_1$ and any vertex $b \in V_2$, let $a = p_0 - p_1 - \cdots - p_k = b$ be a shortest path from a to b in G, and let $P = \{p_0, ..., p_k\}$. Then $H[P]$ is a path, by

Proposition 2.1. Now define $P_X = P \cap X$. Then $H[V_1 \cup V_2 \cup P_X] = H[V_1 \cup V_2 \cup P]$ is a connected induced subgraph of H inducing A_1 and A_2; since $H \in \mathcal{CU}^*$, by Proposition 4.1 we have $H[V_1 \cup V_2 \cup P_X] = H$. Then $V_1 \cup V_2 \cup P_X = V = V_1 \cup V_2 \cup X$, and since $P_X \cap (V_1 \cup V_2) \subseteq X \cap (V_1 \cup V_2) = \emptyset$, we have $X = V \backslash (V_1 \cup V_2) = P_X \subseteq P$. Let q be the smallest natural number such that $p_q \in X$, and let r be the smallest natural number such that $r > q$ and $p_{r+1} \notin X$. Since there are no edges between vertices of V_1 and V_2 (Corollary 4.6), we must have that $p_{q-1} \in V_1$. Now $p_{r+1} \notin X$, so either $p_{r+1} \in V_1$ or $p_{r+1} \in V_2$. If $p_{r+1} \in V_1$ then $H[V_1 \cup V_2 \cup \{p_{r+1}, ..., p_k\})]$ is a connected proper induced subgraph of H inducing A_1 and A_2, a contradiction to Proposition 4.1. If $p_{r+1} \in V_2$ then $H[V_1 \cup V_2 \cup \{p_q, ..., p_r\}]$ is a connected induced subgraph of H inducing A_1 and A_2, so $X = P_X = \{p_q, ..., p_r\}$, and $H[X]$ is a path having endpoint p_q adjacent to some vertex in V_1 and endpoint p_r adjacent to some vertex in V_2. $\square$

Corollary 4.8. *If $H \in \mathcal{CU}^*(A_1, A_2)$ and X is nonempty, then*

$$\left| \left(\bigcup_{v \in V_1} N(v) \right) \cap X \right| = \left| \left(\bigcup_{v \in V_2} N(v) \right) \cap X \right| = 1,$$

and the vertices of X that have neighbors in V_1 and V_2, respectively, are the two ends of the path $H[X]$.

Proof: The result is trivial if $|X| = 1$, so assume $|X| > 1$. We know from Proposition 4.7 that $H[X]$ is a path with its endpoints adjacent to elements of V_1 and V_2, respectively; denote the path by $x_1 - \cdots - x_p$, and suppose without loss of generality that x_1 has a neighbor in V_1, and x_p has a neighbor in V_2. Now suppose that some vertex $u \in V_1$ is adjacent to some path vertex x_i, where $i \neq 1$. Then it is easy to see that $H[V_1 \cup V_2 \cup \{x_i, x_{i+1}, ..., x_p\}]$ is a connected proper induced subgraph of H inducing both A_1 and A_2, a contradiction to the minimality of H. Hence, if any vertex of V_1 has a neighbor in X, that neighbor must be x_1. Similarly, if any vertex of V_2 has a neighbor in X, that neighbor must be x_p. $\square$

Example 4.5. Looking back to Example 4.3, we see that the elements of $\mathcal{CU}^*(S_4, K_3)$ shown illustrate the theorems above: in each graph H, when we choose V_1 and V_2 so that $H[V_1] \cong S_4$ and $H[V_2] \cong K_3$, we have that these vertex sets are disjoint. In the graphs in which X is nonempty, $H[X]$ is a path joining V_1 and V_2. □

These results allow us to conclude a few things about the finiteness of certain $\mathcal{CU}^*$-sets:

Proposition 4.9. *If either A_1 or A_2 is a path, $\mathcal{CU}^*(A_1, A_2)$ is finite.*

Proof: Suppose that A_1 is a path, and that $\mathcal{CU}^*(A_1, A_2)$ is infinite. Then $\mathcal{CU}^*$ certainly contains some graph G such that $|V(G)| \geq 2|V(A_1)| + |V(A_2)|$. Then if we define X, V_1, and V_2 as above, we have that $G[X]$ is isomorphic to a path on at least $|V(A_1)|$ vertices; hence $G[X]$ induces A_1. Then $G[X \cup V_2]$ is a connected proper induced subgraph of G that induces A_1 and A_2, a contradiction to Proposition 4.1. Hence, $\mathcal{CU}^*(A_1, A_2)$ is finite. A similar argument shows that this is also the case if A_2 is a path. □

Proposition 4.10. *Given nonempty graphs A_1, A_2 such that neither graph is induced in the other, if neither A_1 nor A_2 contains a pendant vertex, then $\mathcal{CU}^*(A_1, A_2)$ is infinite.*

Proof: Suppose that neither A_1 nor A_2 contains a pendant vertex, and let G_1 and G_2 be vertex-disjoint graphs isomorphic to A_1 and A_2, respectively. Let r be an arbitrary natural number such that $r \geq \max\{|V(A_1)|, |V(A_2)|\} - 2$, and let $G = (V, E)$, where $V = V(G_1) \cup V(G_2) \cup \{x_1, x_2, ..., x_r\}$ and $E = E(G_1) \cup E(G_2) \cup \{ux_1, x_1x_2, x_2x_3, ..., x_{r-1}x_r, x_rv\}$, where u is an arbitrary vertex of G_1 and v is an arbitrary vertex of G_2. Then any connected induced subgraph on $|V(A_1)|$ vertices is either equal to $G[V(G_1)]$, induced in $G[V(G_2)]$, or contains a pendant vertex. Since

A_1 does not contain a pendant vertex and A_2 does not induce A_1, the only connected induced subgraph of G isomorphic to A_1 is the subgraph induced on $V(G_1)$. Similarly, the only induced subgraph isomorphic to A_2 is the subgraph induced on $V(G_2)$. Then since the vertices $x_1, x_2, ..., x_r$ are clearly cutvertices in G, we have that every proper induced subgraph of G either fails to induced A_1 or A_2 or is disconnected. Then by Proposition 4.1, we conclude that $G \in \mathcal{CU}^*(A_1, A_2)$. Since r was only bounded below, we see then that $\mathcal{CU}^*(A_1, A_2)$ is infinite. $\square$

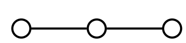
Conjecture 4.11. Given that neither A_1 nor A_2 is induced in the other, then $\mathcal{CU}^*(A_1, A_2)$ is finite if and only if one of A_1, A_2 is a path.

4.2.3 The class $\mathcal{CU}^*(K_n^c)$

We turn our attention now to $\mathcal{CU}^*$-sets for disconnected graphs, beginning with the $\mathcal{CU}^*$-set generated by a single graph, the quintessential disconnected graph K_n^c. As we will see, the graphs in the $\mathcal{CU}^*(K_n^c)$-classes possess a wealth of structure in the way that they can be "put together" to form graphs in $\mathcal{CU}^*(K_n^c)$ for other values of n.

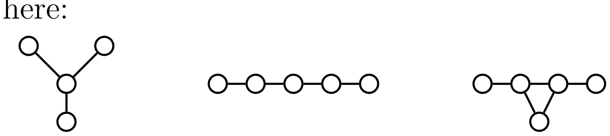
Before beginning, we give some examples for small values of n:

Example 4.6. For the case $n = 2$, we proved in Example 4.1 that $\mathcal{CU}^*(K_2^c)$ consists solely of the graph P_3, shown here:



$\square$

Example 4.7. For the case $n = 3$, we find that $\mathcal{CU}^*(K_3^c) = \{S_4, P_5, \text{bull}\}$. These graphs are shown here:



We note from Proposition 4.1 that each of these graphs belongs to $\mathcal{CU}^*(K_3^c)$. Furthermore, an exhaustive search for elements of $\mathcal{CU}^*(K_3^c)$ among all graphs with less than or equal to 5 vertices (which is easy, though tedious, to perform by hand) yields only these graphs, and, as we shall prove later, any graph in $\mathcal{CU}^*(K_3^c)$ can have at most 5 vertices. Thus, $\mathcal{CU}^*(K_3^c)$ consists exactly of these three graphs. $\square$

We now turn to examining the structure of the graphs in $\mathcal{CU}^*(K_n^c)$ for an arbitrary natural number n.

Lemma 4.12. *For $G \in \mathcal{CU}^*(K_n^c)$, let $W \subseteq V(G)$ be an independent set on n vertices, and let $x \in V(G) \setminus W$ be arbitrary. Then x is a cutvertex of G.*

Proof: The graph $G - x$ is a proper induced subgraph of G that induces K_n^c, so by Proposition 4.1 we must have that $G - x$ is disconnected. $\square$

Corollary 4.13. *If v is an arbitrary vertex of $G \in \mathcal{CU}^*(K_n^c)$ and v is not a cutvertex, then v is an element of every independent set in G containing n vertices.*

Proof: Immediate. $\square$

Lemma 4.14. *Let $G \in \mathcal{CU}^*(K_n^c)$ for $n \geq 2$, let $W \subseteq V(G)$ be an independent subset of order n, and let x be a vertex of G not in W. Then x is adjacent to at least one element of W.*

Proof: Suppose x has no neighbors in W. G has at least two vertices that are not cutvertices, by Proposition 2.3. Let $u \neq x$ denote one of these vertices. By Corollary 4.13, u is in W. Then $W \backslash \{u\} \cup \{x\}$ is an independent set on n vertices in the connected proper induced subgraph $G - u$, a contradiction to Proposition 4.1. Thus, x has at least one neighbor in W. $\square$

Lemma 4.15. *Let $G \in \mathcal{CU}^*(K_n^c)$, let $W \subseteq V(G)$ be an independent set with n vertices, and let x be a vertex of G not in W. Then every component of $G - x$ contains a vertex of W.*

Proof: By Lemma 4.12, $G-x$ is disconnected. Suppose Q is an arbitrary component of $G - x$, and let $v \in V(Q)$ be a vertex of Q at maximum distance from x in G. Since each neighbor of v (with the exception of x, if x and v are adjacent) belongs to the same component as v in $G - x$, it must belong to Q. Then v is such that $d_G(x, v) \geq d_G(x, u)$ for all u adjacent to v. By Proposition 2.5, v is not a cutvertex in G, so by Corollary 4.13, v must belong to W. □

Lemma 4.16. *Given $G \in \mathcal{CU}^*(K_n^c)$, if $x \in V(G)$ is adjacent to a non-cutvertex of G, then for any independent subset $W \subseteq V(G)$ of order n, x is adjacent to at least two vertices of W.*

Proof: Suppose that x is adjacent to non-cutvertex u. Then $u \in W$ for any choice of W, by Corollary 4.13. Now if x is not adjacent to any other vertex in W, then $W \backslash \{u\} \cup \{x\}$ is an independent set on n vertices induced in the connected proper induced subgraph $G - u$ of G, a contradiction to Proposition 4.1. □

Corollary 4.17. *Given $G \in \mathcal{CU}^*(K_n^c)$, if $x \in V(G)$ is adjacent to some pendant vertex v of G then for any independent subset $W \subseteq V(G)$ of order n, x is adjacent to some vertex of $W \backslash \{v\}$.*

Proof: Immediate. □

Lemma 4.18. *Let G be a graph in $\mathcal{CU}^*(K_n^c)$, $W \subseteq V(G)$ an independent set with n vertices, x a vertex of G not in W, and $C_1, C_2, ..., C_q$ the components of $G - x$. For $i = 1, 2, ..., q$, define $n_i = |W \cap V(C_i)|$. Then each C_i contains some vertex v_i adjacent to x in G, and either $C_i \in \mathcal{CU}^*(K_{n_i}^c)$, or v_i is unique and $C_i - v_i \in \mathcal{CU}^*(K_{n_i}^c)$.*

Proof: We show first that each C_i contains some vertex v_i adjacent to x in G. For any C_i and any vertex $a \in V(C_i)$, since G is connected there exists some path $a = p_0 - p_1 - \cdots - p_{k-1} - p_k = x$. Then $v_i = p_{k-1}$ is adjacent to x, and $a - p_1 - \cdots - p_{k-1} = v_i$ is a path joining a and v_i in $G - x$, so v_i belongs to C_i.

Now C_i is a connected component inducing $K^c_{n_i}$, so it induces an element of $\mathcal{CU}^*(K^c_{n_i})$. Let C^*_i denote an induced subgraph of C_i belonging to $\mathcal{CU}^*(K^c_{n_i})$, and let W_i denote an independent subset of $V(C^*_i)$ of order n_i. Let u be a vertex of C^*_i at minimum distance from x, and let $x = q_0 - q_1 - \cdots - q_l = u$ be a shortest path from x to u. Then $G' = G[V(G)\backslash V(C_i) \cup V(C^*_i) \cup \{q_1, ..., q_l\}]$ is a connected induced subgraph of G that induces K^c_n. Since $G \in \mathcal{CU}^*$, we have that $G' = G$. Since $V(C^*_i) \cup \{q_1, ..., q_l) \subseteq V(C_i)$, we have

$$V(C^*_i) \cup \{q_1, ..., q_l\} = V(C_i).$$

Now $q_l \in V(C^*_i)$; if $l = 1$ then $V(C^*_i) = V(C_i)$, and $C_i \in \mathcal{CU}^*(K^c_{n_i})$. Suppose $l > 1$. Then $q_1, ..., q_{l-1}$ are vertices of C_i but not of C^*_i. Note that since these are the intermediate vertices on a shortest path from x to the nearest vertex of $V(C^*_i)$, q_1 is the only vertex from $V(C_i)$ adjacent to x (hence, $v_i = q_1$ and v_i is unique), and q_{l-1} is the only one of these vertices that is adjacent to any vertex of C^*_i. Furthermore, anything the q_j $(1 \leq j \leq l-1)$ are adjacent to (other than x) is by definition a vertex of C_i (since it would lie in this component in $G - x$), and since a shortest path is always an induced path, vertex q_1 is adjacent only to x and q_2 (as long as $l > 2$; otherwise, $q_1 = q_{l-1}$), q_j (where $2 \leq j \leq l-2$) is adjacent only to q_{j-1} and q_{j+1}, and q_{l-1} is adjacent only to q_{l-2} and vertices in C^*_i. Now $W' = W\backslash V(C_i) \cup W_i$ is an independent subset of $V(G)$ of order n. Since $q_1 \notin V(C^*_i)$, we have $q_1 \notin W'$, and since q_1 is adjacent only to x and q_2, by Lemma 4.14 we must have $q_2 \in W'$. Then $l = 2$ in the path above, and we have that $V(C_i) = V(C^*_i) \cup \{q_1\}$. Then

$C_i - v_i = C_i^* \in \mathcal{CU}^*(K_{n_i}^c)$, and the proof is complete. □

Lemma 4.19. *Let G, W, x, and n_i be defined as in Lemma 4.18. If $n_i = 1$ for some i, then $C_i \cong K_1$.*

Proof: By the proof of Lemma 4.18, C_i is either an element of $\mathcal{CU}^*(K_1^c) = K_1$, or $C_i = (\{u, v\}, \{uv\})$, where exactly one vertex of C_i is adjacent to x. Suppose that the latter case holds, and suppose without loss of generality that u is adjacent to x, and v is not. Then v is a pendant vertex and is therefore an element of W. Vertex u must be adjacent to another vertex of W, by Corollary 4.17. Since u is only adjacent to x and v, and $x \notin W$, this is a contradiction. Hence, $C_i \cong K_1$. □

Lemma 4.20. *For $G \in \mathcal{CU}^*(K_n^c)$, given any choice W of n mutually nonadjacent vertices and some $x \in V(G) \backslash W$, each component of $G - x$ contains a vertex that is a pendant vertex in G.*

Proof: By induction on n. The statement is vacuously true for $n = 1$ and also holds for $n = 2$, since $\mathcal{CU}^*(K_2^c) = \{P_3\}$. Suppose now that the statement holds for all natural numbers $n = 1, 2, ..., k$, and let $G \in \mathcal{CU}^*(K_{k+1}^c)$ be arbitrary. Given some choice W of an independent set on $k+1$ vertices, choose arbitrary $x \notin W$. Then $G - x$ is disconnected, by Lemma 4.12, and each component C_i $(1 \leq i \leq q)$ of $G - x$ contains at least one vertex of W (Lemma 4.15), i.e., $n_i \geq 1$ for all i, where $n_i = |W \cap V(G_i)|$. Now by Lemma 4.19, if $n_i = 1$ then $C_i \cong K_1$, and the vertex comprising C_i is a pendant vertex in G, adjacent only to x. If $n_i \geq 2$, then by Lemma 4.18, either $C_i \in \mathcal{CU}^*(K_{n_i}^c)$, or $C_i - v \in \mathcal{CU}^*(K_{n_i})$, where v is adjacent to x in G.

Case: $C_i \in \mathcal{CU}^*(K_{n_i}^c)$. Now $W \cap V(C_i)$ is an independent set on n_i vertices. By the induction hypothesis, if we choose any vertex $x' \notin W \cap V(C_i)$ then C_i has at least one pendant vertex for each component of $C_i - x'$. Now suppose that in G we have

x adjacent to every vertex that is pendant in C_i. We claim that $G - x'$ is connected. Let a, b be any pair of neighbors of x' in G. Then there exist paths from a and b to x that do not include x': If $a \neq x$ (the $a = x$ case is trivial), then the component of $C_i - x'$ that a lies in contains a vertex d that is pendant in C_i. Then there exists a path from a to d that does not include x'; since x is adjacent to every vertex that is pendant in C_i we may append x to this path to get the desired $a - x$ path. The same argument constructs a path $b - x$ not containing x' if $b \neq x$. If we concatenate the paths $a - x$ and $x - b$, identifying the endpoints x, we get a walk from a to b containing an induced path $a - b$ (Proposition 2.2), neither of which contains x'. Then x' is not a cutvertex in G by Proposition 2.4, a contradiction to Lemma 4.12, since $x' \notin W$. We conclude then that x is not adjacent in G to every vertex that is pendant in C_i. But then since no vertex in C_i is adjacent in G to any vertex outside of $V(C_i)$ except for x, there is at least one pendant vertex in C_i that is pendant in G, and our proof is complete for this case.

<u>Case: $C_i - v \in \mathcal{CU}^*(K^c_{n_i})$, where v is adjacent to x in G.</u> We use virtually the same argument as in the previous case. Let $C_i^* = C_i - v$. Then C_i^* contains an independent subset $W_i \subseteq V(C_i^*)$ of order n_i. By the induction hypothesis, if we choose any vertex $x' \notin W_i$ then C_i^* has at least one pendant vertex for each component of $C_i^* - x'$. Now suppose that in G we have v adjacent to every vertex that is pendant in C_i^*. We claim that $G - x'$ is connected. Let a, b be any pair of neighbors of x' in G. Then there exist paths from a and b to v that do not include x': If $a \neq v$ (the case $a = v$ is trivial), then the component of $C_i^* - x'$ that a lies in contains a vertex d that is pendant in C_i^*. Then there exists a path from a to d that does not include x'; since v is adjacent to every vertex that is pendant in C_i^* we may append v to this path to get the desired $a - v$ path. The same argument produces a $b - v$ path if $b \neq v$. If we concatenate the paths $a - v$ and $v - b$, identifying the endpoints v, we get a walk

from a to b containing an induced path $a - b$, neither of which contains x'. Then x' is not a cutvertex in G, a contradiction to Corollary 4.13, since $x' \notin W \backslash V(C_i) \cup W_i$. We conclude then that v is not adjacent in G to every vertex that is pendant in C_i^*. But then since no vertex in C_i^* is adjacent in G to any vertex outside of $V(C_i^*)$ except for v, there is at least one pendant vertex in C_i^* that is pendant in G, and our proof is complete. □

Theorem 4.21. *For arbitrary $G \in \mathcal{CU}^*(K_n^c)$, $|V(G)| \leq 2n - 1$.*

Proof: We prove this by induction on n. Since $\mathcal{CU}^*(K_1^c) = \{K_1\}$, we see from this and Example 4.6 that the result holds for $n = 1$ and 2. Suppose that it also holds for $n = k$, where k is some natural number, and let G be an arbitrary element of $\mathcal{CU}^*(K_{k+1}^c)$. By Lemma 4.20, G has a pendant vertex v. Let x be the neighbor of v in G. Since v is not a cutvertex of G, by Corollary 4.13 it must be included in any choice W of $k + 1$ mutually nonadjacent vertices in G, and by Lemma 4.12 x must be a cutvertex. Let $C_1, C_2, ..., C_q$ denote the components of $G - x$ other than the one containing v, and define $n_i = |W \cap V(C_i)|$. By Lemma 4.18, C_i is either an element of $\mathcal{CU}^*(K_{n_i}^c)$ or has one more vertex than some element of $\mathcal{CU}^*(K_{n_i}^c)$. Then since $n_i < k + 1$, by the induction hypothesis we have that $|V(C_i)| \leq (2n_i - 1) + 1 = 2n_i$. Furthermore, we know from Corollary 4.17 and the proof of Lemma 4.18 that since x is adjacent to a pendant vertex that it must be adjacent to another one of the vertices

w of W, and in the component C_i containing w, we have $C_i \in \mathcal{CU}^*(K_{n_i}^c)$. Then

$$\begin{aligned}|V(G)| &= |\{v,x\}| + \sum_{i=1}^{q} |V(C_i)| \\ &\leq 2 + \left(\sum_{i=1}^{q} 2n_i\right) - 1 \\ &= 1 + 2\sum_{i=1}^{q} n_i \\ &= 1 + 2k \\ &= 2(k+1) - 1,\end{aligned}$$

and by induction the proof is complete. □

We recall that the order of a maximum independent set in G is called the *independence number* of G, and is denoted $\alpha(G)$. Our last result, then, may be restated in the following way: For any graph G with $\alpha(G) \geq n$, there is a connected induced subgraph of G on less than or equal to $2n-1$ vertices with independence number n.

We see that the graphs in $\mathcal{CU}^*(K_n^c)$ have a very clear structure. As we investigate the graphs in $\mathcal{CU}^*(D)$ for an arbitrary disconnected graph D, it becomes apparent that the structure found in $\mathcal{CU}^*(K_n^c)$ owes much to the fact that each component of K_n^c is an isolated vertex; structural theorems similar to the ones above do not hold, in general, for $\mathcal{CU}^*$-sets generated by disconnected graphs. However, the special cases illustrated in this section show that many results concerning the structure and number of the graphs in the class $\mathcal{CU}^*$ may be possible if we limit our attention to specific types of generating graphs for the $\mathcal{CU}^*$-sets.

4.3 Relationships Among $\mathcal{CU}^*$-sets

In this section we state a few results about the relationships among various sets. These results will aid us in our use of $\mathcal{CU}^*$-sets later.

Proposition 4.22. *Let $\mathcal{A} = \{A_1, ..., A_j\}$ be any nonempty collection of graphs, and let $\mathcal{A}' \subseteq \mathcal{A}$ have the property that every graph in $\mathcal{A}$ is induced in some graph of $\mathcal{A}'$. Then $\mathcal{CU}^*(\mathcal{A}') = \mathcal{CU}^*(\mathcal{A})$.*

Proof: Suppose that G is an arbitrary element of $\mathcal{CU}^*(\mathcal{A})$. Then by Proposition 4.1, any connected proper induced subgraph of G fails to induce some $A_i \in \mathcal{A}$. Then it fails to induce some $A_i' \in \mathcal{A}'$, since A_i is induced by at least one element of $\mathcal{A}'$. Then, again by Proposition 4.1, G is an element of $\mathcal{CU}^*(\mathcal{A}')$.

Suppose now that H is an arbitrary element of $\mathcal{CU}^*(\mathcal{A}')$. Then $H \in \mathcal{CU}(A)$, and any connected proper induced subgraph of H fails to induce some element of $\mathcal{A}'$. Since $\mathcal{A}' \subseteq \mathcal{A}$, this induced subgraph fails to induce some element of $\mathcal{A}$, and by Proposition 4.1, we have that $H \in \mathcal{CU}^*(\mathcal{A})$, and the proof is complete. $\Box$

Proposition 4.23. *Say $\mathcal{D} \subseteq \mathcal{C}$, where $\mathcal{C}$ is any nonempty class of graphs. Then*

$$\mathcal{CU}^*(\mathcal{C}) \subseteq \bigcup_{G \in \mathcal{CU}^*(\mathcal{D})} \mathcal{CU}^*\left(\mathcal{C} \backslash \mathcal{D} \cup \{G\}\right).$$

Proof: Let H be an arbitrary element of $\mathcal{CU}^*(\mathcal{C})$. Then H is connected and induces every graph in $\mathcal{D}$, so H induces some graph $G \in \mathcal{CU}^*(\mathcal{D})$. Now since $H \in \mathcal{CU}^*(\mathcal{C})$, every connected proper induced subgraph H' of H fails to induce some graph $F \in \mathcal{C}$. If $F \in \mathcal{D}$ then F is induced by G, and H' does not induce G; otherwise, $F \in \mathcal{C} \backslash \mathcal{D}$ and H' does not induce F. In either case, $H' \notin \mathcal{CU}(\mathcal{C} \backslash \mathcal{D} \cup \{G\})$, and $H \in \mathcal{CU}(\mathcal{C} \backslash \mathcal{D} \cup \{G\})$, so by Proposition 4.1 we have $H \in \mathcal{CU}^*(\mathcal{C} \backslash \mathcal{D} \cup \{G\})$, and our proof is complete. $\Box$

Corollary 4.24. *For any $i \in \{1, ..., j\}$, we have*

$$\mathcal{CU}^*(A_1, ..., A_i, ..., A_j) \subseteq \bigcup_{A' \in \mathcal{CU}^*(A_i)} \mathcal{CU}^*(A_1, ..., A_{i-1}, A', A_{i+1}, ..., A_j).$$

Proof: Immediate. $\Box$

Corollary 4.25.

$$\mathcal{CU}^*(A_1, ..., A_j) \subseteq \bigcup_{A_1' \in \mathcal{CU}^*(A_1)} \cdots \bigcup_{A_j' \in \mathcal{CU}^*(A_j)} \mathcal{CU}^*(A_1', ..., A_j').$$

Proof: As shown in Corollary 4.24,

$$\mathcal{CU}^*(A_1, ..., A_j) \subseteq \bigcup_{A_1' \in \mathcal{CU}^*(A_1)} \mathcal{CU}^*(A_1', A_2, ..., A_j).$$

Now we apply Corollary 4.24 again to obtain

$$\mathcal{CU}^*(A_1, ..., A_j) \subseteq \bigcup_{A_1' \in \mathcal{CU}^*(A_1)} \bigcup_{A_2' \in \mathcal{CU}^*(A_2)} \mathcal{CU}^*(A_1', A_2', A_3, ..., A_j).$$

Continuing inductively in this manner, we obtain the desired result. □

Example 4.8. Suppose we wish to determine the elements of $\mathcal{CU}^*(K_3^c, P_3,$ diamond). By Propositions 4.22 and 4.23 and Example 4.7, we know that

$$\begin{aligned}\mathcal{CU}^*(K_3^c, P_3, \text{diamond}) &= \mathcal{CU}^*(K_3^c, \text{diamond}) \\ &\subseteq \mathcal{CU}^*(P_5, \text{diamond}) \cup \mathcal{CU}^*(S_4, \text{diamond}) \\ &\quad \cup \mathcal{CU}^*(\text{bull, diamond}).\end{aligned}$$

Given any graph in the union above, we can determine if it is an element of $\mathcal{CU}^*(K_3^c, P_3,$ diamond) by applying Proposition 4.1. □

4.4 Recognizing $\mathcal{CU}^*$-sets

Given a collection $G_1, ..., G_k$ of graphs, we wish to determine if there exist graphs $A_1, ..., A_j$ such that $\{G_1, ..., G_k\} = \mathcal{CU}^*(A_1, ..., A_j)$. We approach this problem by first examining the $\mathcal{CU}^*$-sets that each individual G_i belongs to.

Observation 4.26. If G is any connected graph, then $\mathcal{CU}^*(G) = \{G\}$.

Observation 4.27. Let G be any connected graph. Then $G \in \mathcal{CU}^*(G)$, so G always belongs to at least one $\mathcal{CU}^*$-set. Furthermore, since $G \in \mathcal{CU}^*(A_1, ..., A_j)$ implies that G induces each of $A_1, ..., A_j$ and G has only finitely many induced subgraphs, we conclude that G belongs to only finitely many $\mathcal{CU}^*$-sets.

Definition 6. Given a graph G, define $S(G)$ to be the directed graph (V, E), where V is the set of all induced subgraphs of G, and

$$E = \{(G_1, G_2) \mid |V(G_2)| = |V(G_1)| - 1 \text{ and } G_1 \text{ induces } G_2\}.$$

In light of Proposition 4.22, we give the following definition:

Definition 7. A collection $\{A_1, ..., A_j\}$ of graphs is said to be *reduced* if no graph in the collection is induced in any other graph in the list.

Observation 4.28. Every collection $\{A_1, ..., A_j\}$ contains a nonempty reduced subcollection.

We recall that in a directed graph a vertex u is said to be an *ancestor* of a vertex v if there exists a directed path from u to v. We adopt the convention that every vertex is considered an ancestor of itself.

Proposition 4.29. *Let G be any connected graph, and let $G_1, ..., G_k$ be any collection of induced subgraphs of G. Then $G \in \mathcal{CU}^*(G_1, ..., G_k)$ if and only if G is the only connected common ancestor of $G_1, ..., G_k$ in $S(G)$.*

Proof: Suppose $G \in \mathcal{CU}^*(G_1, ..., G_k)$. Then since G induces each of $G_1, ..., G_k$, we have that G is a common ancestor of these vertices in $S(G)$. Now let G' be any connected common ancestor of $G_1, ..., G_k$ in $S(G)$. Then G' is connected and induces each $G_1, ..., G_k$, so G' induces some graph $G'' \in \mathcal{CU}^*(G_1, ..., G_k)$. Now since G' is an induced subgraph of G we have that G'' is as well; since $G \in \mathcal{CU}^*(G_1, ..., G_k)$, by the definition of $\mathcal{CU}^*$ we have that $G'' = G' = G$. Then G is the only connected common ancestor of $G_1, ..., G_k$.

Suppose conversely that G is the only connected common ancestor of $G_1, ..., G_k$. Then no connected proper induced subgraph of G induces each of $G_1, ..., G_k$, so by Proposition 4.1 we have that $G \in \mathcal{CU}^*(G_1, ..., G_k)$. □

Proposition 4.30. *Let G be any connected graph. If there are any two vertices $u, v \in V(G)$ such that $G - u \not\cong G - v$, then $G \in \mathcal{CU}^*(G - u, G - v)$.*

Proof: Since $G - u \not\cong G - v$ and $|V(G - u)| = |V(G - v)|$, we have that neither graph induces the other. Then any common ancestor of the two in $S(G)$ is a graph distinct from each. Since any common ancestor necessarily has a larger vertex set than that of $V(G - u)$ or $V(G - v)$, we conclude that the only common ancestor of $G - u$ and $G - v$ in $S(G)$ is G. By Proposition 4.29, $G \in \mathcal{CU}^*(G - u, G - v)$. □

Proposition 4.31. *Let G be any connected graph. The following are equivalent:*

(1) G belongs to only one $\mathcal{CU}^*$-set having a reduced generating set;

(2) If $\mathcal{C}$ is a reduced collection of graphs and $G \in \mathcal{CU}^*(\mathcal{C})$, then $\mathcal{C} = \{G\}$;

(3) G is vertex-transitive.

Proof: (1) $\iff$ (2): Since G is connected, we know $G \in \mathcal{CU}^*(G)$; the result is immediate.

(2)⇒(3): By Proposition 4.30, we must have that $G - u \cong G - v$ for any vertices $u, v \in V(G)$. It suffices to show that any finite graph having this property is vertex-transitive. This result is not difficult; an outline for its proof is found in [G].

(3)⇒(2): Suppose that G is vertex-transitive. Then for any two vertices $u, v \in V(G)$ there is an automorphism $\varphi : V(G) \to V(G)$ such that $\varphi(u) = v$. Restricting φ to $V(G)\backslash\{u\}$, we see that $G - u \cong G - v$ for any vertices $u, v \in V(G)$. Then since

G has at least one non-cutvertex (Proposition 2.3), $G - v$ must be connected for all v. Now it is clear that in $S(G)$, $G - v$ is an ancestor to everything G is, except for G itself. If G is to be the only connected common ancestor of all graphs in a class $\mathcal{C}$, as in Proposition 4.29, we must have $G \in \mathcal{C}$. Then if $\mathcal{C}$ is a reduced collection of graphs we have that $\mathcal{C} = \{G\}$. This proves (2) and completes our proof. $\square$

Now given any graphs $G_1, ..., G_k$, in order for $\{G_1, ..., G_k\}$ to be a subset of $\mathcal{CU}^*(A_1, ..., A_j)$ for some graphs $A_1, ..., A_j$, clearly $G_i \in \mathcal{CU}^*(A_1, ..., A_j)$ for $i = 1, ..., k$. This suggests that we can determine the $\mathcal{CU}^*$-sets that $\{G_1, ..., G_k\}$ is a subset of by identifying the $\mathcal{CU}^*$-set possibilities common to each G_i. By explicitly finding the elements of these common $\mathcal{CU}^*$-sets, we can determine whether or not $\{G_1, ..., G_k\}$ is *equal* to any $\mathcal{CU}^*$-set.

5 Forbidden Subgraph Characterizations

We now resume our consideration of the class $\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k)$ and its forbidden subgraph characterizations. Just as in the case of $\mathcal{CU}^*$-sets, it will be instructive to consider this problem in cases depending on the connectivity of the graphs involved.

5.1 The Connected Case

We start by considering the case in which $H_1, H_2, ..., H_k$ are all nonempty connected graphs. Our result follows immediately from what we have done previously.

Theorem 5.1. *Given arbitrary nonempty connected graphs $H_1, ..., H_k$, we have*

$$\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k) = \mathcal{G}\big(\mathcal{CU}^*(H_1, H_2, ..., H_k)\big).$$

Proof: Suppose that $G = G_1 \cup G_2 \cup \cdots \cup G_k$, where G_i is an arbitrary element of $\mathcal{G}(H_i)$ for $i = 1, 2, ..., k$. If a graph in $\mathcal{CU}^*(H_1, H_2, ..., H_k)$ is induced in G, since any graph in $\mathcal{CU}^*$ is connected it would have to be induced entirely within one component of some G_i, and G_i would then induce H_i, a contradiction. Hence,

$$\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k) \subseteq \mathcal{G}\big(\mathcal{CU}^*(H_1, H_2, ..., H_k)\big).$$

Now suppose that G is a graph in $\mathcal{G}\big(\mathcal{CU}^*(H_1, H_2, ..., H_k)\big)$. Let G_1 be the induced subgraph of G composed of all components of G not inducing H_1. Continuing inductively, let G_i be the induced subgraph of G composed of all components of G not inducing H_i and not already contained in G_j for some $j < i$. Note that for any i, G_i may be the empty graph. Now since $G_i \in \mathcal{G}(H_i)$ for each i, if $G = G_1 \cup \cdots \cup G_k$ then $G \in \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$. If $G \neq G_1 \cup \cdots \cup G_k$, then there exists some component C of G such that C contains each of $H_1, H_2, ..., H_k$. But then C belongs to

$\mathcal{CU}(H_1, H_2, ..., H_k)$, and so has an induced subgraph that lies in $\mathcal{CU}^*$, a contradiction. Then

$$\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k) \supseteq \mathcal{G}\big(\mathcal{CU}^*(H_1, H_2, ..., H_k)\big),$$

and the proof is complete. $\square$

We have thus given a forbidden subgraph characterization for the case in which each H_i is connected. Furthermore, we claim that this is the "best possible" forbidden subgraph characterization, in that no proper subset of $\mathcal{CU}^*(H_1, ..., H_k)$ suffices to characterize $\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k)$. To see this, let $\mathcal{C}$ denote a proper subset of $\mathcal{CU}^*(H_1, ...H_k)$, and let F be an arbitrary element of $\mathcal{CU}^*(H_1, ..., H_k) \backslash \mathcal{C}$. Then F is not an element of $\mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k)$, but $F \in \mathcal{G}(\mathcal{C})$, since by definition no element of $\mathcal{CU}^*$ induces any other element of $\mathcal{CU}^*$. Then $\mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k) \neq \mathcal{G}(\mathcal{C})$. Our claim is thus proved; the forbidden subgraph characterization given above is minimal with respect to the number of the forbidden subgraphs involved.

Example 5.1. We recall from Example 4.2 that $\mathcal{CU}^*(P_3, K_3) = \{\text{paw, diamond}\}$. This tells us

$$\mathcal{G}(P_3) \oplus \mathcal{G}(K_3) = \mathcal{G}(\text{paw, diamond}).$$

From this we conclude that we can partition the components of any graph G that is (paw, diamond)-free into one induced subgraph that is P_3-free, and hence (as we will show in Section 6.1) the union of any number of complete components, and another induced subgraph that is K_3-free. Conversely, any graph that is the union of any number of complete graphs and any number of triangle-free graphs must be (paw, diamond)-free. $\square$

5.2 The Disconnected Case

5.2.1 Preliminaries

Finding a forbidden subgraph characterization for $\mathcal{G}(H_1)\oplus\mathcal{G}(H_2)\oplus\cdots\oplus\mathcal{G}(H_k)$ in the case where one or more of the H_i is disconnected is considerably more difficult than in the case where each H_i is connected. We begin by defining a few terms and stating some preliminary results. In the following subsections we consider the consequences of these results in various cases.

Definition 8. Given the graph class $\mathcal{G} = \mathcal{G}(H_1)\oplus\mathcal{G}(H_2)\oplus\cdots\oplus\mathcal{G}(H_k)$ and an arbitrary graph F, F is said to have a *free partition* $(F_1, ..., F_k)$ *(with respect to* $(H_1, ..., H_k)$*)* if $F = F_1 \cup \cdots \cup F_k$ and each F_i is an element of $\mathcal{G}(H_i)$.

A free partition is thus a partition of the components of F into k (possibly empty) subgraphs. Note that F may have more than one free partition.

Observation 5.2. $F \in \mathcal{G}$ if and only if a free partition exists for F.

Corollary 5.3. *Suppose that F is a minimal forbidden subgraph for $\mathcal{G} = \mathcal{G}(H_1)\oplus\mathcal{G}(H_2)\oplus\cdots\oplus\mathcal{G}(H_k)$, i.e., F is forbidden for $\mathcal{G}$, but no proper induced subgraph of F is forbidden. Then for any component Q of F, $F - Q = F[V(F)\backslash V(Q)]$ has a free partition.*

Proof: Immediate from Observation 5.2. $\square$

Just as in the case for connected graphs considered in Section 5.1, we would like to find *minimal* forbidden subgraphs, those without any extraneous vertices or components. For if $\mathcal{F}$ denotes the class of all forbidden subgraphs for a hereditary class $\mathcal{C}$ of graphs, and if $\mathcal{F}' \subseteq \mathcal{F}$ denotes the subset of all minimal forbidden subgraphs, then it is easy to see that $\mathcal{C} = \mathcal{G}(\mathcal{F}) = \mathcal{G}(\mathcal{F}')$.

Example 5.2. It is not difficult to see, and we will prove later, that

$$\mathcal{G}(K_2^c) \oplus \mathcal{G}(K_2) = \mathcal{G}(P_3, 2K_2),$$

and that the minimal forbidden subgraphs for the class $\mathcal{G}(K_2^c) \oplus \mathcal{G}(K_2)$ are P_3 and $2K_2$. $\square$

We now develop results that will aid us in finding minimal forbidden subgraphs for the class $\mathcal{G} = \mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k)$. In each of the following propositions, let $\mathcal{G} = \mathcal{G}(H_1) \oplus \mathcal{G}(H_2) \oplus \cdots \oplus \mathcal{G}(H_k)$, and let F be a minimal forbidden subgraph for $\mathcal{G}$.

Proposition 5.4. *The graph F induces each H_i, for $i = 1, ..., k$.*

Proof: Suppose that F does not induce H_j for some $j \in \{1, ..., k\}$. Then $(\emptyset, ..., \emptyset, F, \emptyset, ..., \emptyset)$ is a free partition of F, where F is in the jth position and $\emptyset$ denotes the empty graph. However, F is forbidden and so does not have a free partition. We conclude that F induces H_i for each i. $\square$

Proposition 5.5. *For any component Q of F and any $i \in \{1, ..., k\}$, Q induces a component of H_i.*

Proof: Suppose that component Q of F does not induce any component of H_j. Now by Corollary 5.3, $F' = F - Q$ has a free partition $(F'_1, ..., F'_k)$. Then $F = F' \cup Q$ has a free partition $(F'_1, ..., F'_{j-1}, F'_j \cup Q, F'_{j+1}, ..., F'_k)$, a contradiction, since F is a forbidden subgraph for $\mathcal{G}$. We conclude that Q induces some component of H_i for all i. $\square$

Corollary 5.6. *If H_i is connected, then H_i is induced in every component of F.*

Proof: Immediate. $\square$

Proposition 5.7. *If Q is any component of F and $\mathcal{C}_i$ denotes the set of all induced subgraphs of H_i induced in Q $(i = 1, ..., k)$, then $Q \in \mathcal{CU}^*(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k)$.*

Proof: By definition Q is a connected graph which induces each element of $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$. Suppose now that Q' is a connected proper induced subgraph of Q. Then since F is minimal, $F' = (F - Q) \cup Q'$ is not a forbidden subgraph for $\mathcal{G}$ and thus has a free partition $(F'_1, ..., F'_k)$. Suppose Q' is a component of F'_i. Then $(F'_1, ..., F'_{i-1}, F'_i - Q', F'_{i+1}, ..., F'_k)$ is a free partition of $F' - Q'$, and hence of $F - Q$. Now $F = (F' - Q') \cup Q$ has no free partition, so $(F'_i - Q') \cup Q$ induces H_i. However, $F'_i = (F'_i - Q') \cup Q'$ does not induce H_i. Then Q contains some induced subgraph of H_i (and hence some element of $\mathcal{C}_i$) that Q' does not. Then Q' fails to induce some element of $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$. Since Q' was an arbitrary connected proper induced subgraph of Q, we conclude from Proposition 4.1 that $Q \in \mathcal{CU}^*(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k)$. $\square$

Proposition 5.8. *If F is connected, then $F \in \mathcal{CU}^*(H_1, ..., H_k)$. Furthermore, each graph in $\mathcal{CU}^*(H_1, ..., H_k)$ is a forbidden subgraph for $\mathcal{G}$, though it is not necessarily minimal.*

Proof: It is clear that each graph in $\mathcal{CU}^*(H_1, ..., H_k)$ is forbidden for $\mathcal{G}$, since such a graph has no free partition. Suppose now that F is connected. By Proposition 5.4, F induces each H_i. Then by Proposition 5.7, $F \in \mathcal{CU}^*(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k)$, where $\mathcal{C}_i$ here consists of all induced subgraphs of H_i. By Proposition 4.22 we see that $\mathcal{CU}^*(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k) = \mathcal{CU}^*(H_1, ..., H_k)$, and the proof is complete. $\square$

5.2.2 One Disconnected Graph

We will see, particularly in the next subsection, that determining *exactly* which graphs constitute a complete list of minimal forbidden subgraphs for $\mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$, where at least one H_i is disconnected, appears extraordinarily difficult, in general, due to the large number of ways each disconnected H_i can "fit" into a minimal forbidden subgraph F. However, in the case where only one H_i is disconnected we can come closer to achieving this exact list of minimal forbidden subgraphs by specifying a relatively small subset of the forbidden subgraphs of $\mathcal{G}$ which contains all minimal forbidden subgraphs.

Theorem 5.9. *Suppose that H_1 is disconnected and that $H_2, ..., H_k$ are all connected. Let $\mathcal{P}$ denote the class of partitions $\Delta = \{P_1, ..., P_n\}$ of the components of H_1 into nonempty subgraphs of H_1. Then if $\mathcal{F}$ is the set of minimal forbidden subgraphs for $\mathcal{G}$, we have*

$$\mathcal{F} \subseteq \bigcup_{\Delta=\{P_1,...,P_n\}\in\mathcal{P}} \mathcal{CU}^*(P_1, H_2, ..., H_k) \oplus \cdots \oplus \mathcal{CU}^*(P_n, H_2, ..., H_k).$$

Furthermore, each of the graphs in the above union is a forbidden subgraph of $\mathcal{G}$.

Proof: Let F be an arbitrary element of $\mathcal{F}$, and let $Q_1, ..., Q_q$ denote the components of F. Now for any component Q_i of F, we know that $F - Q_i$ has a free partition, and since each component of F induces $H_2, ..., H_k$ (by Corollary 5.6), this free partition must be $(F - Q_i, \emptyset, ..., \emptyset)$, where $\emptyset$ denotes the empty graph. Then $F - Q_i$ does not induce H_1, but F does. Now let $V_H \subseteq V(F)$ be a set of vertices such that $F[V_H] \cong H_1$. Then since $F - Q_i$ is H_1-free for all i, we must have that $V_H \cap V(Q_i) \neq \emptyset$ for all $i = 1, ..., q$. Then $P_i = Q_i[V_H \cap V(Q_i)]$ is a nonempty union of components of H_1, induced in Q_i. Then $Q_i \in CU(P_i, H_2, ..., H_k)$, and we claim that $Q_i \in \mathcal{CU}^*(P_i, H_2, ..., H_k)$. For if this is not the case then Q_i contains a connected proper induced subgraph $Q_i' \in \mathcal{CU}^*(P_i, H_2, ..., H_k)$. But then $F' = (F - Q_i) \cup Q_i'$ is a proper induced subgraph of F that has no free partition and is thus forbidden, a contradiction to the minimality of F. Then F belongs to the class $\mathcal{CU}^*(P_1, H_2, ..., H_k) \oplus \cdots \oplus \mathcal{CU}^*(P_q, H_2, ..., H_k)$, and we have proved the first assertion of the theorem.

Now let G be any element of

$$\bigcup_{\Delta=\{P_1,...,P_n\}\in\mathcal{P}} \mathcal{CU}^*(P_1, H_2, ..., H_k) \oplus \cdots \oplus \mathcal{CU}^*(P_n, H_2, ..., H_k).$$

Then, similar to the argument above, since every component of G induces $H_2, ..., H_k$, if G were to have a free partition it would have to be $(G, \emptyset, ..., \emptyset)$, where $\emptyset$ denotes the empty graph. However, by construction we know that G induces H_1, so G has no free partition and must therefore be a forbidden subgraph for $\mathcal{G}$. $\square$

Let us illustrate this theorem with some examples:

Example 5.3. The class $\mathcal{G} = \mathcal{G}(K_2^c) \oplus \mathcal{G}(K_2)$ consists of the graphs of the form $K_n \cup mK_1$ for any nonnegative integers m, n. We wish to find the minimal forbidden subgraphs for $\mathcal{G}$. We note that the only partitions of the components of K_2^c are of the forms $\Delta_1 = \{K_2^c\}$ and $\Delta_2 = \{K_1, K_1\}$. By Theorem 5.9, the set $\mathcal{F}$ of minimal forbidden subgraphs for $\mathcal{G}$ is a subset of

$$\mathcal{CU}^*(K_2^c, K_2) \cup \big(\mathcal{CU}^*(K_1, K_2) \oplus \mathcal{CU}^*(K_1, K_2)\big).$$

Simplifying this expression, we obtain that $\mathcal{F} \subseteq \{P_3, 2K_2\}$, and we easily verify that both P_3 and $2K_2$ are minimal forbidden subgraphs for $\mathcal{G}$. Then since $\mathcal{G} = \mathcal{G}(\mathcal{F})$, we have that

$$\mathcal{G}(K_2^c) \oplus \mathcal{G}(K_2) = \mathcal{G}(P_3, 2K_2),$$

and we have proved the assertion made in Example 5.2. $\square$

Example 5.4. We wish to find the minimal forbidden subgraphs for the class $\mathcal{G} = \mathcal{G}(K_2^c) \oplus \mathcal{G}(K_3)$. By Theorem 5.9, they are among the elements of

$$\mathcal{CU}^*(K_2^c, K_3) \cup \big(\mathcal{CU}^*(K_1, K_3) \oplus \mathcal{CU}^*(K_1, K_3)\big),$$

and by noting that $\mathcal{CU}^*(K_2^c) = \{P_3\}$ and applying Corollary 4.24, we see that this set is a subset of

$$\mathcal{CU}^*(P_3, K_3) \cup \big(\mathcal{CU}^*(K_1, K_3) \oplus \mathcal{CU}^*(K_1, K_3)\big),$$

which equals

$$\{\text{paw},\ \text{diamond}, 2K_3\}$$

by Example 4.2, Proposition 4.22, and Observation 4.26. We can also verify that each of these graphs is a minimal forbidden subgraph of $\mathcal{G}$. Then

$$\mathcal{G}(K_2^c) \oplus \mathcal{G}(K_3) = \mathcal{G}(\text{paw, diamond, } 2K_3).$$

$\square$

5.2.3 Two or More Disconnected Graphs

We have shown thus far that if F is a minimal forbidden subgraph for $\mathcal{G} = \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$, then F induces each H_i, and every component of F induces at least one component of each H_i. We have shown that each component of F belongs to a $\mathcal{CU}^*$-set generated by the induced subgraphs of each H_i that are induced in the component. In the case where no H_i was disconnected we determined exactly what the minimal forbidden subgraphs were, and in the case where exactly one H_i was disconnected we determined a small subset of the forbidden subgraphs that contained the minimal forbidden subgraphs. As we allow more of the H_i to be disconnected, however, the number of cases to check in finding minimal forbidden subgraphs grows very rapidly, making it difficult to efficiently discover the minimal forbidden subgraphs.

We demonstrate some of the complexity of this problem in the case where at least two of the H_i are disconnected. Suppose that $H_1, ..., H_d$ $(d > 1)$ are disconnected and $H_{d+1}, ..., H_k$ are connected, and let F be a minimal forbidden subgraph of $\mathcal{G} = \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$. If F has one component, Proposition 5.8 tells us that $F \in \mathcal{CU}^*(H_1, ..., H_k)$. Now suppose that F has exactly two components Q_1 and Q_2. Then $F - Q_1$ has a free partition $(\emptyset, ..., \emptyset, Q_2, \emptyset, ..., \emptyset)$, where $\emptyset$ denotes the empty graph. Suppose the Q_2 in this free partition lies in the jth element. Then since F has no free partition we must have that Q_1 induces all H_i for $i \neq j$. Furthermore, Q_1 does not induce H_j, since if it did it would be a forbidden subgraph for $\mathcal{G}$, and F would not be minimal. However, $Q_1 \cup Q_2$ induces H_j. Now $F - Q_2$ also has a free partition. Since Q_1 induces all H_i for $i \neq j$, we must have that this free partition is $(\emptyset, ..., \emptyset, Q_1, \emptyset, ..., \emptyset)$, where $\emptyset$ denotes the empty graph and Q_1 is located in the jth position. Then by symmetry, we have that Q_2 induces each H_i for $i \neq j$, and Q_2 does not induce H_j. By the same reasoning as in Theorem 5.9, if $\mathcal{P}_i$ denotes the set of partitions of the components of H_i into two nonempty subgraphs, we have that the minimal forbidden subgraphs of $\mathcal{G}$ having two components all belong to the set

$$\bigcup_{i=1}^{d} \left(\bigcup_{\{P_1, P_2\} \in \mathcal{P}_i} \mathcal{CU}^*(H_1, ..., H_{i-1}, P_1, H_{i+1}, ..., H_k) \oplus \mathcal{CU}^*(H_1, ..., H_{i-1}, P_2, H_{i+1}, ..., H_k) \right).$$

This set already shows signs of being more complicated than the similarly-formed set in the case where $d = 1$. Experimenting with free partitions of component-deleted forbidden subgraphs on more than two components convinces us that this problem is certainly not trivial—the number of unions and direct summands involved in expressions similar to that above rapidly increases, with few apparent patterns—and we do not undertake its solution here. Happily, our results in the case where at most one of the H_i is disconnected will allow us to derive some useful results, which we present in the next section.

6 Further Results

6.1 The Class $\mathcal{G}(\mathcal{CU}^*(H))$

Suppose we wished to find the forbidden subgraph characterization for $\mathcal{G}(H) \oplus \mathcal{G}(H)$. If H is connected this is trivial; Theorem 5.1 tells us that this is precisely $\mathcal{G}(\mathcal{CU}^*(H)) = \mathcal{G}(H)$. In other words, the union of two H-free graphs is H-free, if H is connected. However, if H is disconnected, what can $\mathcal{G}(\mathcal{CU}^*(H))$ tell us?

Proposition 6.1. *The class of arbitrary unions of H-free graphs is exactly the class $\mathcal{G}(\mathcal{CU}^*(H))$.*

Proof: Suppose $G = G_1 \cup \cdots \cup G_n$, where each G_i is H-free. We note in passing that G itself may not be H-free; for example, K_1 is K_3^c-free, but $K_1 \cup K_1 \cup K_1$ is not. However, if G induces any element of $\mathcal{CU}^*(H)$ then there is a component of G that induces H, a contradiction. Hence, G is $\mathcal{CU}^*(H)$-free.

Suppose conversely that F is an arbitrary element of $\mathcal{G}(\mathcal{CU}^*(H))$. Then for any component Q of F, if Q induces H then it induces some element of $\mathcal{CU}^*(H)$. Since this is not true, Q is H-free, and we can write F as the union of H-free graphs. $\square$

Example 6.1. We have already seen and used the fact that $\mathcal{CU}^*(K_2^c) = \{P_3\}$ in previous sections. In light of Proposition 6.1, we see that $\mathcal{G}(K_2^c) \oplus \cdots \oplus \mathcal{G}(K_2^c) = \mathcal{G}(P_3)$, i.e., the class of arbitrary unions of complete graphs is exactly the class of P_3-free graphs. $\square$

6.2 Joins

With the results that we have obtained so far, we can say a few things about a related problem. We recall from graph theory that the *join* of graphs $G_1, G_2, ..., G_k$ is denoted by $G_1 \vee G_2 \vee \cdots \vee G_k$ and is the graph (V, E), where $V = \bigcup_{i=1}^{k} V(G_i)$ and

$$E = \bigcup_{i=1}^{k} E(G_i) \cup \{uv \mid u \in G_i, v \in G_j, i \neq j\}.$$

Observation 6.2.

$$G_1 \vee \cdots \vee G_k = (G_1^c \cup \cdots \cup G_k^c)^c.$$

Proposition 6.3. *Given arbitrary graphs G and H, G is H-free if and only if G^c is H^c-free.*

Proof: G induces $H \iff$ there exists some subset V' of $V(G)$ such that $G[V'] \cong H$ $\iff G^c[V'] \cong H^c \iff G^c$ induces H^c. $\square$

Proposition 6.4. *A graph G is complete multipartite if and only if G is $(K_2 \cup K_1)$-free.*

Proof: Using Proposition 6.3, Example 6.1, and the definition of complete multipartite graphs, we see that G is complete multipartite if and only if G^c is a union of complete subgraphs if and only if G^c is P_3-free if and only if G is $(K_2 \cup K_1)$-free. $\square$

Let $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$ be nonempty classes of nonempty graphs, and let

$$\mathcal{G}(\mathcal{H}_1) \vee \mathcal{G}(\mathcal{H}_2) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k)$$

denote the class of all graphs G that can be expressed in the form $G_1 \vee G_2 \vee \cdots \vee G_k$, where $G_i \in \mathcal{G}(\mathcal{H}_i)$ for $i = 1, 2, ..., k$. Then we can use our results from the previous section to obtain some information about $\mathcal{G}(\mathcal{H}_1) \vee \mathcal{G}(\mathcal{H}_2) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k)$.

Theorem 6.5. *If $\mathcal{H}_1, ..., \mathcal{H}_k$ are arbitrary nonempty classes of nonempty graphs, then the class $\mathcal{G}(\mathcal{H}_1) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k)$ has a forbidden subgraph characterization.*

Proof: It is clear from Observation 6.2 that $\mathcal{G}(\mathcal{H}_1) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k)$ consists of the complements of the graphs in $\mathcal{G}(\mathcal{H}_1^c) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k^c)$, where $\mathcal{H}_i^c$ denotes the class of graphs whose complement is contained in $\mathcal{H}_i$. We have shown already in Theorem 3.2 that $\mathcal{G}(\mathcal{H}_1^c) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k^c)$ has a forbidden subgraph characterization, so

$$\mathcal{G}(\mathcal{H}_1^c) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k^c) = \mathcal{G}(\mathcal{F})$$

for some collection $\mathcal{F}$ of graphs. Then by Proposition 6.3, we have that $\mathcal{G}(\mathcal{H}_1) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k) = \mathcal{G}(\mathcal{F}^c)$, where $\mathcal{F}^c$ denotes the set of graphs whose complements lie in $\mathcal{F}$. Hence $\mathcal{G}(\mathcal{H}_1) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k)$ has a forbidden subgraph characterization. □

Proposition 6.6. *If $H_1, H_2, ..., H_k$ all have connected complements, then*

$$\mathcal{G}(H_1) \vee \mathcal{G}(H_2) \vee \cdots \vee \mathcal{G}(H_k) = \mathcal{G}\big(\mathcal{CU}^*(H_1^c, ..., H_k^c)^c\big),$$

where $\mathcal{CU}^*(H_1^c, ..., H_k^c)^c$ denotes the set of complements of graphs in $\mathcal{CU}^*(H_1^c, ..., H_k^c)$.

Proof: The graph G is an element of $\mathcal{G}(H_1) \vee \mathcal{G}(H_2) \vee \cdots \vee \mathcal{G}(H_k)$ if and only if $G^c \in \mathcal{G}(H_1^c) \oplus \mathcal{G}(H_2^c) \oplus \cdots \oplus \mathcal{G}(H_k^c) = \mathcal{G}(\mathcal{CU}^*(H_1^c, H_2^c, ..., H_k^c))$, by Proposition 6.3 and Theorem 5.1. Taking complements in both classes, we obtain the desired result. □

6.3 Classes with Multiple Forbidden Subgraphs

Thus far we have considered in depth the class $\mathcal{G} = \mathcal{G}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$ where each $\mathcal{H}_i$ consists of a single nonempty graph H_i. Now we examine the class $\mathcal{G}$ when each $\mathcal{H}_i$ may contain more than one graph.

Proposition 6.7. *For any collection $\mathcal{H}$ of graphs, $\mathcal{G}(\mathcal{H}) = \bigcap_{H \in \mathcal{H}} \mathcal{G}(H)$.*

Proof: Let G be an arbitrary element of $\mathcal{G}(\mathcal{H})$. Then for each $H \in \mathcal{H}$ we have that G does not induce H, and hence $G \in \mathcal{G}(H)$. Then $G \in \bigcap_{H \in \mathcal{H}} \mathcal{G}(H)$.

Suppose conversely that $G \in \bigcap_{H \in \mathcal{H}} \mathcal{G}(H)$. Then G is H-free for every graph in $\mathcal{H}$, so $G \in \mathcal{G}(\mathcal{H})$. □

Proposition 6.8. *If $\mathcal{H}_1, ..., \mathcal{H}_k$ are arbitrary nonempty classes of nonempty graphs, then*

$$\mathcal{G}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k) = \bigcap_{H_1 \in \mathcal{H}_1} \cdots \bigcap_{H_k \in \mathcal{H}_k} \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k).$$

Proof: Suppose $G \in \mathcal{G}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$. Then we can write $G = G_1 \cup \cdots \cup G_k$, where $G_i \in \mathcal{G}(\mathcal{H}_i)$ for $i = 1, ..., k$. Then by Proposition 6.7, $G_i \in \bigcap_{H_i \in \mathcal{H}_i} \mathcal{G}(H_i)$ for all i. Then $G = G_1 \cup \cdots \cup G_k \in \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$ for every $H_1 \in \mathcal{H}_1, ..., H_k \in \mathcal{H}_k$, and hence $G \in \bigcap_{H_1 \in \mathcal{H}_1} \cdots \bigcap_{H_k \in \mathcal{H}_k} \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$.

Suppose conversely that $G \in \bigcap_{H_1 \in \mathcal{H}_1} \cdots \bigcap_{H_k \in \mathcal{H}_k} \mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$. Then for any component Q of G, suppose that Q induces a graph $H_i' \in \mathcal{H}_i$ for $i = 1, ..., k$. Then certainly $G \notin \mathcal{G}(H_1') \oplus \cdots \oplus \mathcal{G}(H_k')$, a contradiction, so for any component Q of G, we have $Q \in \mathcal{G}(\mathcal{H}_j)$ for some $j \in \{1, ..., k\}$. We write $G = G_1 \cup \cdots \cup G_k$, where G_1 consists of all components that are $\mathcal{H}_1$-free, G_2 consists of all components that are $\mathcal{H}_2$-free and not in $\mathcal{H}_1$, G_3 consists of all components that are $\mathcal{H}_3$-free and not in $\mathcal{H}_1 \cup \mathcal{H}_2$, and so on up through G_k. Then $G_i \in \mathcal{G}(\mathcal{H}_i)$ for all i, so $G \in \mathcal{G}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$, and our proof is complete. □

This result allows us to reduce the general case of finding a forbidden subgraph characterization of $\mathcal{G}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k)$ to finding forbidden subgraph characterizations of various $\mathcal{G}(H_1) \oplus \cdots \oplus \mathcal{G}(H_k)$, where H_i is a single graph in $\mathcal{H}_i$. We will demonstrate its use of these results in the next section.

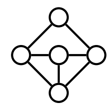
6.4 A Forbidden Subgraph Problem

We turn now to the problem that inspired our study. Barrett and Loewy have shown the following [BL]:

Theorem 6.9. *The following are equivalent:*

(1) G is a graph of the form $(K_m \cup K_n \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$, where $k, m, n, p_i, q_i, r \geq 0$ for all i;

(2) G is a $(P_4, \mathrm{paw} \cup K_1, \mathrm{diamond} \cup K_1, 3K_3, K_{2,2,2}, \widehat{W}_4)$-free graph, where $\widehat{W}_4$ denotes the graph shown here:



We will give a proof of this theorem using the machinery we have developed throughout this thesis. First we establish a few results.

Proposition 6.10. *G is a union of at most two nonempty complete subgraphs if and only if G is (P_3, K_3^c)-free.*

Proof: We saw in Example 6.1 that a graph G is a union of complete subgraphs if and only if G is P_3-free. Now if G is the union of at most two nonempty complete subgraphs then certainly K_3^c is not induced in G; if G is the union of three or more nonempty complete subgraphs then K_3^c is induced. □

Proposition 6.11. $\mathcal{CU}^*(K_2 \cup K_1) = \{P_4, \mathrm{paw}\}$.

Proof: By Proposition 4.1, both P_4 and the paw graph are elements of $\mathcal{CU}^*(K_2 \cup K_1)$. Now let G be a P_4-free element of $\mathcal{CU}^*(K_2 \cup K_1)$, and let $u, v, w \in V(G)$ be vertices such that $G[\{u, v, w\}] \cong K_2 \cup K_1$ and vw is an edge in G. Suppose without loss of generality that $d(u, v) \leq d(u, w)$. Since G is connected, there is a shortest path from u to v. This path will not contain w. Now since G is P_4-free, this path cannot have length greater than 2. Hence there exists a vertex x such that $u - x - v$ is a path in G. Now since $\{u, x, v, w\}$ does not induce P_4, we must have x adjacent to w. Then $\{u, v, w, x\}$ induces the paw graph. Since G is in $\mathcal{CU}^*(K_2 \cup K_1)$ and G induces the paw, by definition of $\mathcal{CU}^*$ we must have that $G = \mathrm{paw}$, and the proof is complete. □

Proposition 6.12. *G is an arbitrary union of complete bipartite graphs if and only if G is (P_4, K_3)-free.*

Proof: Suppose G is an arbitrary union of complete bipartite graphs. Then G is a union of connected complete multipartite graphs. By Proposition 6.4, applied to the components of G, we have that each component of G is $(K_2 \cup K_1)$-free and is hence P_4-free. Then G is also P_4-free. Since G is bipartite, G is K_3-free as well; hence G is (P_4, K_3)-free.

Suppose G is (P_4, K_3)-free. Then G induces no odd cycles, and is therefore bipartite. Furthermore, since the paw induces K_3, we have that G is (P_4, paw)-free and is thus, by Propositions 6.11 and 6.4, a union of complete multipartite graphs. Then G is a union of complete bipartite graphs. □

Corollary 6.13. *The set of graphs of the form described in statement (1) of Theorem 6.9 is the class $\big(\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)\big) \vee \mathcal{G}(K_2^c)$.*

Proof: Immediate from Propositions 6.10 and 6.12. □

Proof of the Theorem: In light of Corollary 6.13, our problem is to show that

$$\big(\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)\big) \vee \mathcal{G}(K_2^c) = \mathcal{G}(P_4, \text{paw} \cup K_1, \text{diamond} \cup K_1, 3K_3, K_{2,2,2}, \widehat{W_4}).$$

We accomplish this in stages.

STAGE 1: Show that $\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3) = \mathcal{G}(P_4, \text{paw}, \text{diamond}, 3K_3)$.

By Proposition 6.8,

$$\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3) = \big(\mathcal{G}(P_3) \oplus \mathcal{G}(P_4)\big) \cap \big(\mathcal{G}(P_3) \oplus \mathcal{G}(K_3)\big) \\ \cap \big(\mathcal{G}(K_3^c) \oplus \mathcal{G}(P_4)\big) \cap \big(\mathcal{G}(K_3^c) \oplus \mathcal{G}(K_3)\big). \quad (1)$$

Now we also have that

$$\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3) = \mathcal{G}(\mathcal{F}),$$

where $\mathcal{F}$ is some collection of graphs, by Theorem 3.2. We assume that $\mathcal{F}$ consists of only minimal forbidden subgraphs. We now determine what those forbidden subgraphs are. Slightly modifying the statement of Proposition 6.7, though following the same reasoning, we have that *each forbidden subgraph for any of the classes listed in the intersection is also a forbidden subgraph for the intersection.* Furthermore, the collection of graphs which are minimal forbidden subgraphs for at least one of the classes in the intersection suffices as a list of forbidden subgraphs of the intersection. Then to find minimal forbidden subgraphs for the intersection we need only determine which of the minimal forbidden subgraphs for each of the classes involved in the intersection are minimal for the intersection.

First, we find the class $\mathcal{G}(P_3) \oplus \mathcal{G}(P_4)$. By Theorem 5.1, Proposition 4.22, and Observation 4.26,

$$\mathcal{G}(P_3) \oplus \mathcal{G}(P_4) = \mathcal{G}(\mathcal{CU}^*(P_3, P_4)) = \mathcal{G}(\mathcal{CU}^*(P_4)) = \mathcal{G}(P_4).$$

Then a graph belongs to $\mathcal{G}(P_3) \oplus \mathcal{G}(P_4)$ if and only if it is P_4-free. We see that P_4 is in fact a minimal forbidden subgraph for $\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)$; any proper induced subgraph of it belongs to the class.

Next, we find the class $\mathcal{G}(P_3) \oplus \mathcal{G}(K_3)$. Again by Theorem 5.1 and Example 4.2,

$$\mathcal{G}(P_3) \oplus \mathcal{G}(K_3) = \mathcal{G}(\mathcal{CU}^*(P_3, K_3)) = \mathcal{G}(\text{paw}, \text{diamond}).$$

So a graph belongs to $\mathcal{G}(P_3) \oplus \mathcal{G}(K_3)$ if and only if it is (paw, diamond)-free. One checks easily that both of these are minimal forbidden subgraphs for $\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)$.

We now turn our attention to the class $\mathcal{G}(K_3^c) \oplus \mathcal{G}(P_4)$. We can find the minimal forbidden subgraphs for this class by applying the results of Section 5.2. However, we know from Proposition 5.4 that any minimal forbidden subgraph for this class will induce P_4. Such graphs, since they cannot equal P_4 (P_4 does not induce K_3^c, as these graphs must) will not be minimal forbidden subgraphs for $\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)$, so it is unneccesary to determine them.

Turning now to the class $\mathcal{G}(K_3^c) \oplus \mathcal{G}(K_3)$, we know from Theorem 5.9 that the minimal forbidden subgraphs for this class are all elements of the set

$$\begin{aligned} &\mathcal{CU}^*(K_3^c, K_3) \,\cup\, \big(\mathcal{CU}^*(K_2^c, K_3) \oplus \mathcal{CU}^*(K_1, K_3)\big) \\ &\qquad\qquad\qquad \cup\, \big(\mathcal{CU}^*(K_1, K_3) \oplus \mathcal{CU}^*(K_1, K_3) \oplus \mathcal{CU}^*(K_1, K_3)\big). \qquad (2) \end{aligned}$$

Now if $G \in \mathcal{CU}^*(K_3^c, K_3)$ then $G \in \mathcal{CU}(K_2^c, K_3)$. G then induces an element of $\mathcal{CU}^*(K_2^c, K_3)$, which by Corollary 4.24 is a subset of $\mathcal{CU}^*(P_3, K_3) = \{\text{paw}, \text{diamond}\}$. Since the paw and the diamond are already minimal forbidden subgraphs for $\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)$, we conclude that no new minimal forbidden subgraphs will be gained from $\mathcal{CU}^*(K_3^c, K_3)$.

Moving on to the next class of graphs in (2), we note that

$$\begin{aligned} \mathcal{CU}^*(K_2^c, K_3) \oplus \mathcal{CU}^*(K_1, K_3) &\subseteq \mathcal{CU}^*(P_3, K_3) \oplus \mathcal{CU}^*(K_3) \\ &= \{\text{paw, diamond}\} \oplus \{K_3\}. \end{aligned}$$

Then again we have that each graph in this collection induces either the paw or the diamond; thus, $\mathcal{CU}^*(K_2^c, K_3) \oplus \mathcal{CU}^*(K_1, K_3)$ provides us with no new minimal forbidden subgraphs for $\mathcal{G}(P_3, K_3^c) \oplus \mathcal{G}(P_4, K_3)$.

The last class of graphs in (2) is the class

$$\begin{aligned}\mathcal{CU}^*(K_1,K_3)\oplus\mathcal{CU}^*(K_1,K_3)\oplus\mathcal{CU}^*(K_1,K_3) &= \mathcal{CU}^*(K_3)\oplus\mathcal{CU}^*(K_3)\oplus\mathcal{CU}^*(K_3)\\ &= \{K_3\}\oplus\{K_3\}\oplus\{K_3\}\\ &= \{3K_3\}.\end{aligned}$$

Now any proper induced subgraph of $3K_3$ is an induced subgraph of $2K_3\cup K_2$, which has a free partition of the form $(2K_3,K_2)$. Then since $3K_3$ is forbidden, it is a minimal forbidden subgraph for $\mathcal{G}(P_3,K_3^c)\oplus\mathcal{G}(P_4,K_3)$.

Now since we have inspected all of the forbidden subgraphs for each of the classes in (1) and found that they all induce either P_4, paw, diamond, or $3K_3$, and since the minimal forbidden subgraphs of $\mathcal{G}(P_3,K_3^c)\oplus\mathcal{G}(P_4,K_3)$ are all found among the minimal forbidden subgraphs of each of the classes listed in (1), we conclude that

$$\mathcal{G}(P_3,K_3^c)\oplus\mathcal{G}(P_4,K_3)=\mathcal{G}(P_4,\text{paw},\ \text{diamond},3K_3),$$

and Stage 1 of our proof is complete.

STAGE 2: Show that

$$\mathcal{G}(P_4,\text{paw},\text{diamond},3K_3)\vee\mathcal{G}(K_2^c)=\mathcal{G}(P_4,\text{paw}\cup K_1,\text{diamond}\cup K_1,3K_3,K_{2,2,2},\widehat{W}_4)$$

Since most of our theory thus far has dealt with direct sums of graph classes, and the discussion in Section 6.2 relates joins to direct sums, it will be more convenient for us to consider the complements of the graphs above and find a forbidden subgraph characterization for the class $\mathcal{G}(P_4,P_3\cup K_1,K_2\cup 2K_1,K_{3,3,3})\oplus\mathcal{G}(K_2)$. By Proposition 6.8, this class is equal to the class

$$\begin{aligned}&\big(\mathcal{G}(P_4)\oplus\mathcal{G}(K_2)\big)\ \cap\ \big(\mathcal{G}(P_3\cup K_1)\oplus\mathcal{G}(K_2)\big)\\ &\qquad\cap\ \big(\mathcal{G}(K_2\cup 2K_1)\oplus\mathcal{G}(K_2)\big)\ \cap\ \big(\mathcal{G}(K_{3,3,3})\oplus\mathcal{G}(K_2)\big).\end{aligned}\tag{3}$$

We now proceed through this collection of classes as we did in Stage 1. First we consider $\mathcal{G}(P_4) \oplus \mathcal{G}(K_2)$. It is clear from Theorem 5.1 that this class is equal to $\mathcal{G}(P_4)$, and P_4 is in fact a minimal forbidden subgraph of $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$.

Next, we know the minimal forbidden subgraphs of $\mathcal{G}(P_3 \cup K_1) \oplus \mathcal{G}(K_2)$ are among the elements of

$$\mathcal{CU}^*(P_3 \cup K_1, K_2) \;\cup\; \big(\mathcal{CU}^*(P_3, K_2) \oplus \mathcal{CU}^*(K_1, K_2)\big). \tag{4}$$

Now say $G \in \mathcal{CU}^*(P_3 \cup K_1, K_2) = \mathcal{CU}^*(P_3 \cup K_1)$. Since we are looking for minimal forbidden subgraphs of $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$ and we know that P_4 is one, let us suppose that G is P_4-free. Now let $\{u, v, w, x\}$ be vertices of G such that $G[\{u, v, w, x\}] \cong P_3 \cup K_1$, with $u - v - w$ a path. Now since G is connected, there exists some shortest path from x to a member of $\{u, v, w\}$. Since G does not induce P_4, this path has only one intermediate vertex y. It is a simple matter to check that if any vertex of $\{u, v, w\}$ is not adjacent to y, then G induces a P_4. Hence, y is adjacent to each of u, v, w, and x, and $G[\{u, v, w, x, y\}] \cong$ dart. We can verify that the dart is a minimal forbidden subgraph for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$, and since every P_4-free graph of $\mathcal{CU}^*(P_3 \cup K_1)$ induces the dart, this class offers us no additional minimal forbidden subgraphs.

Considering now the set

$$\begin{aligned} \mathcal{CU}^*(P_3, K_2) \oplus \mathcal{CU}^*(K_1, K_2) &= \mathcal{CU}^*(P_3) \oplus \mathcal{CU}^*(K_2) \\ &= \{P_3\} \oplus \{K_2\} \\ &= \{P_3 \cup K_2\} \end{aligned}$$

in (4), we verify also that $P_3 \cup K_2$ is a minimal forbidden subgraph for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$.

Moving to the next class in (3), we wish to find any minimal forbidden subgraphs among the forbidden subgraphs of $\mathcal{G}(K_2 \cup 2K_1) \oplus \mathcal{G}(K_2)$. We know that the minimal forbidden subgraphs for this class are among the elements of

$$\begin{aligned} \mathcal{CU}^*(K_2 \cup 2K_1, K_2) \;\cup\; &\big(\mathcal{CU}^*(K_2, K_2) \oplus \mathcal{CU}^*(2K_1, K_2)\big) \\ &\cup\; \big(\mathcal{CU}^*(K_2 \cup K_1, K_2) \oplus \mathcal{CU}^*(K_1, K_2)\big) \\ &\qquad \cup\; \big(\mathcal{CU}^*(K_2, K_2) \oplus \mathcal{CU}^*(K_1, K_2) \oplus \mathcal{CU}^*(K_1, K_2)\big). \end{aligned} \tag{5}$$

We consider each of the classes in this union in turn.

We begin with the class $\mathcal{CU}^*(K_2 \cup 2K_1, K_2) = \mathcal{CU}^*(K_2 \cup 2K_1)$. Since P_4 is already known to be a minimal forbidden subgraph for the intersection in (3), and our goal is to find the minimal forbidden subgraphs for this intersection, let us suppose that G is a P_4-free element of this $\mathcal{CU}^*$-set. Suppose further that $u, v, w, x \in V(G)$ are such that $G[\{u, v, w, x\}] \cong K_2 \cup 2K_1$, with uv an edge. Since G is connected, there exists a shortest path from w to u, and since G is P_4-free, this path has only one intermediate vertex $y \neq v$. Now since we do not induce P_4 on $\{w, y, u, v\}$, we must have y adjacent to v. Similarly, there exists a vertez z such that $x - z - u$ is a path and z is adjacent to v. Now suppose that there is no vertex in G that u,v,w, and x are all adjacent to. Then $z \neq y$, and wz and xy are not edges. Then either yz is not an edge, in which case $\{w, y, u, z\}$ induces P_4, or yz is an edge, in which case $\{w, y, z, x\}$ induces P_4. This contradiction leads us to conclude that there is a vertex that is a common neighbor of u, v, w, and x. Without loss of generality suppose y is a such a vertex. Then $\{u, v, w, x, y\}$ induces the $\ltimes$ graph, illustrated in Section 2. One checks easily that $\ltimes \in \mathcal{CU}^*(K_2 \cup 2K_1)$, so in fact $G = \ltimes$, and also that $\ltimes$ is a minimal forbidden subgraph for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$.

The next class of graphs in (5) is

$$\begin{aligned}\mathcal{CU}^*(K_2, K_2) \oplus \mathcal{CU}^*(2K_1, K_2) &= \mathcal{CU}^*(K_2) \oplus \mathcal{CU}^*(2K_1, K_2)\\ &\subseteq \{K_2\} \oplus \mathcal{CU}^*(P_3, K_2)\\ &= \{K_2\} \oplus \mathcal{CU}^*(P_3)\\ &= \{K_2\} \oplus \{P_3\}\\ &= \{P_3 \cup K_2\}.\end{aligned}$$

Since we have already found $P_3 \cup K_2$ to be a minimal forbidden subgraph for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$, we gain no new minimal forbidden subgraphs from this class.

Applying Proposition 6.11, we see that the next class of graphs in (5) is

$$\begin{aligned}\mathcal{CU}^*(K_2 \cup K_1, K_2) \oplus \mathcal{CU}^*(K_1, K_2) &= \mathcal{CU}^*(K_2 \cup K_1) \oplus \mathcal{CU}^*(K_2)\\ &= \{P_4, \text{paw}\} \oplus \{K_2\}\\ &= \{P_4 \cup K_2, \text{paw} \cup K_2\}.\end{aligned}$$

Since each of these graphs induces $P_3 \cup K_2$, we see that this class gives us no new minimal forbidden subgraphs for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$.

Finishing our list in (5), we consider

$$\begin{aligned}\mathcal{CU}^*(K_2, K_2) \oplus \mathcal{CU}^*(K_1, K_2) \oplus \mathcal{CU}^*(K_1, K_2) &= \mathcal{CU}^*(K_2) \oplus \mathcal{CU}^*(K_2) \oplus \mathcal{CU}^*(K_2)\\ &= \{K_2\} \oplus \{K_2\} \oplus \{K_2\}\\ &= \{3K_2\}.\end{aligned}$$

Since any proper induced subgraph of $3K_2$ is an induced subgraph of $2K_2 \cup K_1$, which has a free partition of the form $(2K_2, K_1)$, we see that $3K_2$ is in fact a minimal forbidden subgraph for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$.

Returning to consider the last class of graphs in (3), we look at $\mathcal{G}(K_{3,3,3}) \oplus \mathcal{G}(K_2)$. By Theorem 5.1, this class is exactly

$$\mathcal{CU}^*(K_{3,3,3}, K_2) = \mathcal{CU}^*(K_{3,3,3}) = \{K_{3,3,3}\}.$$

Once again it is easy to see that $K_{3,3,3}$ is a minimal forbidden subgraph for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$.

Recalling our argument from Stage 1, we know that the minimal forbidden subgraphs for the intersection in (3) are found among the minimal forbidden subgraphs for each of the classes involved in the intersection. We therefore collect the information from the paragraphs above and conclude that a complete list of minimal forbidden subgraphs for $\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2)$ is $\{P_4$, dart, $P_3 \cup K_2, \ltimes, 3K_2, K_{3,3,3}\}$, and hence

$$\mathcal{G}(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3}) \oplus \mathcal{G}(K_2) = \mathcal{G}(P_4, \mathrm{dart}, P_3 \cup K_2, \ltimes, 3K_2, K_{3,3,3}).$$

Applying Proposition 6.3, we see that

$$\mathcal{G}(P_4, \mathrm{paw}, \mathrm{diamond}, 3K_3) \vee \mathcal{G}(K_2^c) = \mathcal{G}(P_4, \mathrm{paw} \cup K_1, \widehat{W}_4, \mathrm{diamond} \cup K_1, K_{2,2,2}, 3K_3),$$

and hence Stage 2 of our proof is complete. Combining the results of Stages 1 and 2, our proof of Theorem 6.9 is complete. $\square$

7 Conclusion

In the course of this thesis we have proved the existence of a forbidden subgraph characterization for the classes

$$\mathcal{G}(\mathcal{H}_1) \oplus \cdots \oplus \mathcal{G}(\mathcal{H}_k) \tag{6}$$

and

$$\mathcal{G}(\mathcal{H}_1) \vee \cdots \vee \mathcal{G}(\mathcal{H}_k), \tag{7}$$

and we have developed results that are useful in determining what such forbidden subgraph characterizations are. We have applied our results to Theorem 6.9, and found them to be sufficient to exactly determine the forbidden subgraphs in a characterization problem. One will note that our proof of Theorem 6.9 is considerably longer and more tedious than the one found in [BL]. The main advantage to our approach is that it provides a systematic graph-theoretic approach to finding the necessary minimal forbidden subgraphs, whereas the methods used by the authors cited include techniques from outside of graph theory which may not be available in all problems of this type. Our results provide us with alternatives to trial and error in finding the minimal forbidden subgraphs, and guarantee a complete list of them.

However, many more results that will simplify and answer problems similar to that of Barrett and Loewy await discovery and proof. For example, our knowledge of the properties of $\mathcal{CU}^*$-sets allowed us to simplify formulations of graph classes and even perform a sort of arithmetic on them, as demonstrated in Section 6.4. What more can we discover about these sets? When is a given $\mathcal{CU}^*$-set guaranteed to be finite or infinite? We recall from Section 4 that every vertex in X (as defined there) is a cutvertex of the graph $G \in \mathcal{CU}^*$ it belongs to. What can the study of connectivity tell us about the graphs in $\mathcal{CU}^*$, especially the study of graphs having a specified number of *non*-cutvertices?

Yet another line of inquiry to be pursued is that of reversing our process: Throughout the thesis we took classes such as (6) and (7) and characterized them in terms of a collection of forbidden subgraphs. Under what conditions is it possible to express a class having a forbidden subgraph characterization as a class of the form (6) or (7)? In the case of Barrett and Loewy's problem, statement (1) of Theorem 6.9 is by far the easier of the two formulations to use in algorithmically determining if a graph belongs to the class described. It seems reasonable that expressions of the form (6) and (7) may be easier to work with, in general.

Furthermore, the study of $\mathcal{CU}^*$-sets, particularly as taken up in Section 4.4, bears a strong resemblance to certain facets of the Reconstruction Conjecture, which states that every graph G is uniquely determined by the collection of its vertex-deleted graphs $G-v$. Our $\mathcal{CU}^*$-sets appear to have much to do with the *legitimate deck* problem: Which collections of graphs can constitute the set of induced subgraphs of order $|V(G)|-1$ for some graph G? In the terms of our problem, which collections of graphs $G_1, ..., G_k$, where the vertex of each G_i has order $k-1$, are such that $\mathcal{CU}^*(G_1, ..., G_k)$ has an element with a vertex set of order k?

Thus, it is apparent that much can be done to extend the results of this thesis, especially in the study of $\mathcal{CU}^*$-sets. Definitive results in these areas might have bearing on current research and provide many applications in various graph-theoretical problems.

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